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Equilibrium yield curves under regime switching*

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Abstract: This paper studies how inflation as a macroeconomic indicator affects nominal bond prices. I consider an economy with a representative agent with Epstein- Zin preferences. Regime switching affects the state-space capturing inflation and consumption growth. Thus, the agent is concerned about the intertemporal distribution of risk, which is affected by the persistence of the variables and the regimes. Regime switching allows for structural changes in the volatility of unexpected shocks. To the extent that inflationary unexpected shocks indicate lower consumption growth, nominal bond holders need to be compensated for these shocks. It follows that a switch in the regime state affecting the covariance of inflation and consumption growth can be interpreted as a change in the price of risk. I find coefficients of risk aversion from 40 to 90, and subjective discount factors above 0.99, depending on the exact specification of the model. The model yields have on average a positive slope, a consistent Principal Components decomposition, and predictability as in Cochrane and Piazzesi (2002).
Keywords: Consumption-based Asset Pricing; Regime Switching; Recursive Preferences; Yield Curve; Term Structure of Interest Rates.
JEL Classification: G12, E42, E43, E44, E31.

Resumen: Este documento estudia cómo la inflación como un indicador macroeconómico afecta a los precios de los bonos nominales. Considero una economía con un agente representativo que tiene preferencias Epstein-Zin. El espacio-estado que captura la inflación y el crecimiento en el consumo, se ve afectado por cambios en regímenes. Así al agente le preocupa la distribución intertemporal del riesgo, que es afectada por la persistencia de las variables y de los regímenes. Los regímenes permiten capturar cambios estructurales en la volatilidad de los choques inesperados. En la medida que los choques inflacionarios inesperados indiquen un menor crecimiento en el consumo, los tenedores de los bonos nominales serán compensados por estos choques. Por lo anterior, un cambio en el régimen que afecta a la covarianza de los choques a la inflación y al crecimiento en el consumo, puede ser interpretado como un cambio en el precio del riesgo. Encuentro coeficientes de aversión al riesgo entre 40 y 90, y factores de descuentos subjetivos mayores a 0.99, dependiendo de la especificación del modelo. Las tasas de interés del modelo tienen en promedio una pendiente positiva, una descomposición de Componentes Principales consistente, y predictibilidad como en Cochrane y Piazzesi (2002).
Palabras Clave:Valuación de Activos basada en el Consumo; Cambios en Regímenes; Preferencias Recursivas; Curva Cupón Cero; Estructura Temporal de Tasas de Interés.

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1 Introduction

This paper studies the role of inflation as a macroeconomic indicator affecting bond prices. I consider an economy with a representative agent with Epstein-Zin preferences. Regime switching affects the state-space capturing inflation, and consumption growth. Thus, the agent is concerned about the intertemporal distribution of risk, which is affected by the persistence of the variables and the regimes.

Consider an unexpected increase in inflation, it affects the nominal yields in two ways. Directly, by decreasing the real component of the nominal yields. Indirectly, by increasing the nominal yields to the extent it is a prelude of lower consumption growth. The latter effect is a compensation for nominal bond holders. Additionally, the relationship between inflation, and consumption is subject to regime switching. Thus, a switch in the regime state affecting the covariance of inflation, and consumption growth can be interpreted as a change in the price of risk.

Having Epstein-Zin preferences, the agent is concerned about the intertemporal distribution of risk. Intuitively, he cares about the persistence in the changes of the variables, and of the regimes. Thus, understanding the temporality of the changes, in the variables, and in the regimes, is central to assess their effects on the yield curve.

Regime switching in the model is central to the implied behavior of yields. First, a regime switching in the variance-covariance matrix of the shocks impinging consumption growth and inflation, implies heteroscedasticity in the yields, and a time-varying risk premium, a documented feature of the data. Second, regime switching allows for structural changes in the volatility of unexpected shocks. To the extent inflationary unexpected shocks indicate lower consumption growth, nominal bond holders need to be compensated, as explained. Thus, a switch in the regime state affecting the covariance of inflation and consumption growth can be interpreted as a change in the price of risk. Third, a regime switch in the mean inflation broadly translates to a regime in the level of nominal yields. Post-war yields data have gone through structural changes in the past decades, particularly during the 1970s and 1980s. The regimes are key to explain these periods.

The model rests on three central assumptions. First, inflation provides a predictable component to consumption growth. Inflation then plays a double role in the model, it not only allows to obtain the prices of nominal bonds but it also provides a predictable component. Second, regime switching affects the model capturing the dynamics of inflation, and consumption growth. Its presence accounts for the time-varying dynamics in these variables. Third, the representative agent has Epstein-Zin preferences. Under these preferences the persistence of the variables, and of the regimes are of special concern to the agent.

In contrast to the more common time-separable preferences, under recursive preferences the agent cares about the intertemporal distribution of risk. The persistence of the variables, and of the regime states affect how the agent perceives the intertemporal distribution of risk. Under time-separable preferences the agent is risk averse but indifferent to persistence. While under recursive preferences, the agent is not only risk averse but also has a taste for persistence. The persistence coupled with recursive preferences is key to obtain variability in the Stochastic Discount Factor (SDF).

There are three fundamental steps in how I proceed. First, by assuming an elasticity of intertemporal substitution equal to one, I obtain close solutions for the prices of bonds and for yields. Second, I estimate the model using Maximum Likelihood and Gibbs-sampling procedures, exploiting both methods to perform statistical tests and to address measurement issues. Third, a common procedure in the literature is to posit the regimes directly into the SDF. In contrast, I assume that the regime switching affects the model capturing inflation and consumption growth. This *implies* regime switching in the SDF, allowing for a more structural model.

I find that coefficients of risk aversion from 40 to 90, and subjective discount factors above 0.99, depending on the exact specification of the model can capture central stylized facts. First, the model has an average upward sloping nominal yield curve and higher variability in the long end of the yield curve. In contrast, consumption-based models with time separable preferences imply a downward sloping nominal yield curve and have low variability in the long end of the yield curve, something not observed in the data. In the model, the average nominal yield curve slopes upward because the yields in the long end have a higher compensation for the intertemporal distribution of risk relative to those in the short end.

Second, I decompose the implied yields of the model using Principal Component Analysis as in Litterman and Scheinkman (1991). I compare them to the decomposition of the yields data, and show that they have a consistent Principal Components decomposition. An analysis of the model's dynamics suggests why the Principal Component decomposition for the yields model is consistent.

Third, I study deviations from the Expectation Hypothesis (EH) or, equivalently, predictability in the yields. There are models that are successful in accounting for the predictability in yields by introducing regimes switching, e.g. see Banzal and Zhou (2001). Yet, their regime switching is introduced directly in the SDF, and their state variables are modeled as latent with no explicit macroeconomics exogenous variables. The model's yields have predictability as seen in the data in terms of the Cochrane and Piazzesi (2002) tent-shaped function of forwards. The regimes coupled with Epstein-Zin preferences are able to account for this feature.

There are at least three potential issues with the model. First, the extent to which one is able to measure the predictability component of consumption growth given by inflation. To address this issue, I consider the probability distribution of the estimated parameters to assess the measurement errors, and run tests on the consumption growth time series to measure the relative importance of its random walk constituent. Second, how regimes affect the dynamics exactly. To deal with this point, I motivate various specifications of the model, discuss their relative merits, and perform statistical tests to compare them against each other. Third, the model uses calculations of elements in the long run. By their nature these are hard to measure. I discuss aspects of the statistical accuracy of these measurements.

Understanding how macroeconomic variables affect asset prices is central to finance and macroeconomics. Both subjects have an interest in understanding the macroeconomic sources, the amounts, and the prices of risk. Macroeconomic, or aggregate risk, should be the only risk priced. Thus, a promising line of research is studying additional measures of risk, such as the intertemporal distribution of risk, as functions of macroeconomic variables, and its implications on asset prices. I draw from Hansen (2006), Hansen (2007), Hansen, Heaton and Li (2008), and to build on Piazzesi and Schneider (2006), adding regime switching to the fundamentals.[1] This accounts for the time-varying dynamics in the macroeconomic variables. The model maintains, as in Piazzesi and Schneider (2006), the upward slope in the nominal yields, and additionally accounts for a number of stylized facts.

[1]Three central differences between this paper and Piazzesi and Schneider (2006) are the use of regime switching, the subjective discount factor is smaller than one, and the infinite horizon.

2 Literature Review

Consumption-based asset pricing has had its ebb and flow for the past decades in the economic literature. The floodgates were open with papers like Lucas (1978), Hall (1978), Mehra and Prescott (1985), Hansen and Singleton (1982), (1983), Campbell and Shiller (1991), Cox, Ingersoll, and Ross (1985), among others. The flow came back with papers like Constantinides (1990), Abel (1990), Epstein and Zin (1991), Hansen and Jagannathan (1991), Campbell and Shiller (1991), Constantinides and Duffie (1996), and Campbell and Cochrane (1999). The current tide is formed by papers like Lettau and Ludwigson (2001), Wachter (2006), Piazzesi, Schneider, and Tuzel (2007), Bansal and Yaron (2004), Parker and Julliard (2005), Hansen, Heaton, and Li (2008), among many others.

This paper draws from Hansen, Heaton and Li (2008), Hansen (2006), Hansen (2007), and Hansen (2008), in order to build on Piazzesi and Schneider (2006). Compared to Piazzesi and Schneider (2006), it examines other issues, uses regimes, and differs on its methodology. A central difference between Piazzesi and Schneider (2006), and Bansal and Yaron (2004) is that while consumption growth process is estimated in the former, it is calibrated in the latter. Piazzesi and Schneider (2006) and this paper rely on being able to have some consumption growth predictability through inflation. Parker and Julliard (2005), is similar in that they argue that cumulative consumption growth has a predictable component, which is compatible with the results of the VR tests in this paper.[2] For the predictability of yields papers like Banzal and Zhou (2001) explore the implications. They have had success using regimes to model the predictability in yields.[3] In comparison, this paper introduces the regime switching to the state-space capturing the dynamics of consumption growth and inflation.

The discrete-time (affine) Term Structure Models in the literature have three conceptually different approaches. The first, as in this paper, has a parametric specification for the SDF.[4] The second one, starts positing a continuous SDF and then uses the discrete time version of the continuous process. This approach has the property that as the time interval gets smaller, the models converge to their continuous counterparts. The third, is similar to the second one in that it starts with a continuous process but approximates it. For this, it uses, for

[2]See subsection 10 for the definition of the test.

[3]Although, their models's SDF can be motivated with a general equilibrium model; they introduce regimes directly to the SDF, and assume latent variables to calibrate their model directly to yields data.

[4]These can be further subdivided into consumption-based, factor based, and latent variable based. The frontier among them blurs and some models incorporate elements of the others.

example, the Euler expansion (see Glasserman (2003)). The choice is a matter of aims, convenience, and tractability.

A standard reference for the empirical effects of monetary policy in the long run is McCandless and Weber (1995), topic for which there is less disagreement. Regarding the effects on the short run there is much more variety, which reflects the fact that there is less consensus on the matter. Some central papers are Lucas (1972) and Calvo (1983). On the empirical estimates of the effects of monetary policy on the short run, the following are representative sources: Friedman and Schwartz (1971), a classic, Sims and Zha (2006) and Christiano, Eichenbaum and Evans (1994). Relevant to methodology is, for example, Eichembaum (1992). More on the structural side there is Rotemberg and Woodford (1998) as a prominent example. Alternative ways of measuring monetary shocks are given in Romer and Romer (2003), and Boschen and Mills (1995). One notable model that relates monetary policy to the behavior of the term structure of interest rates is Piazzesi (2005). The literature in these areas is vast and I have only named a quite limited number of papers.

3 The model

Consider an endowment economy with a representative agent that has Epstein-Zin preferences. Prices adjust so that the agent maximizes utility given his endowment. The exogenous consumption and inflation processes are modeled with the following state-space:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_{\Delta c} + x_{1,t} + \epsilon_{1,t+1} \qquad (1)\\
\pi_{t+1} &= \mu_{\pi}(s_{1,t}) + x_{2,t} + \epsilon_{2,t+1}\\
\text{where}&\\
\mathbf{x}_{t+1} &= (x_{1,t} \quad x_{2,t})',\\
\mathbf{x}_{t+1} &= \phi_x \mathbf{x}_t + K\epsilon_{t+1},
\end{aligned}$$

c_t denotes the logarithm of consumption and π_t denotes inflation at time t. The mean consumption growth $\mu_{\Delta c}$ is a scalar. The shock $\epsilon_{t+1} \equiv (\epsilon_{1,t+1} \quad \epsilon_{2,t+1})'$ affecting *all* equations is normally distributed with mean $(0 \ \ 0)'$, and the variance-covariance matrix $\Omega(s_{2,t})$. The vector $\mathbf{x}_t$ is latent. ϕ_x and K are 2×2 matrices. The matrix ϕ_x needs to have the absolute value of its eigenvalues strictly less than 1 for $\mathbf{x}_t$ to be stationary random variables. I sometimes stack Δc_{t+1} and π_{t+1} in the vector: $\mathbf{z}_{t+1} \equiv (\Delta c_{t+1} \;\; \pi_{t+1})'$.

The mean inflation $\mu_\pi(s_{1,t})$, a scalar, and the matrix $\Omega(s_{2,t})$ are, respectively, subject to regime switching with two regime states each. The regimes $s_{1,t}$, $s_{2,t}$ follow Markov chains with transition probability matrices Q and R, respectively,

$$Q \equiv \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix}, \qquad R \equiv \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}.$$

$q_{i,j} \equiv Pr[s_{1,t+1} = j | s_{1,t} = i]$ and $r_{i,j} \equiv Pr[s_{2,t+1} = j | s_{2,t} = i]$ are the probabilities of switching in one period to regime state j given that the chain is in regime state i, respectively.[5] For notation purposes $\mathbf{s}_t \equiv (s_{1,t}, s_{2,t})$. These regimes are assumed to have different relationships, e.g. $s_{1,t}$ fixed, perfect correlation (i.e. $s_{1,t} = s_{2,t}$), etc.[6] This is the baseline specification, later on the exact specifications and extensions are described.[7]

Inflation provides a predictable component to consumption growth through $\mathbf{x}_t$. Had consumption growth been modeled as a random walk, the Epstein-Zin preferences and the constant relative risk aversion (CRRA) preferences would have been observational equivalent (Kocherlakota (1990)).

Changes in the regime state in $\mu_\pi(s_{1,t})$ are interpreted as changes in inflationary regimes.[8] They can also be interpreted as changes in monetary regimes. While this is a broad interpretation, as typically other parameters might be recognized as being affected by changes in monetary regimes, it allows to sidestep the problem of deciding which is the best indicator of a monetary regime change.[9] Also, as it is well understood, there is a lag between changes in monetary policy and its effects on inflation.

The persistence of $\mu_\pi(s_{1,t})$ can be measured by the expected time in a regime

[5] For a more general relationship between the two, we can define a new regime: $\mathbf{s}_t \equiv (s_{1,t}, s_{2,t})$ and thus, $M[i,j] = m_{i,j} \equiv Pr[\mathbf{s}_{t+1} = j | \mathbf{s}_t = i]$ can be directly constructed. Note that under independence $M = Q \otimes R$, which I am assuming for one specification of the model.

[6] It is common in the literature to associate regimes states with booms and recessions, in particular in those models where the product is a state variable. This interpretation does not hold in the model presented in this paper.

[7] The state-space can be seen as a generalization of an AR(1) process. Note that if $K = \phi_x$ and regimes are fixed, it becomes an AR(1) and, under stationarity, can also be written as a MA(∞) process.

[8] Inflationary regimes have been identified in Evans and Lewis (1995) and Evans and Wachtel (1993).

[9] For example, Sims and Zha (2006) consider different specifications of a macroeconomic model to study regime switching. Another example, in the specific case of the monetary policy variable Bernanke and Blinder (1992) have argued that the best indicator has changed through time. Recent monetary research (see for example, see Woodford in Beyer and Reichlin (2008)) has advocated for having inflation directly as the policy variable.

state, $1/(1-q_{i,i})$, measured in quarters. While the persistence of ϵ_t in $\mathbf{x}_t$ is captured by the matrix ϕ_x and can be measured by the half-life of a shock, or by the magnitude of the eigenvalues of ϕ_x. In this sense, the transition matrix Q and ϕ_x keep a parallelism. Accordingly, changes, say, in $x_{2,t}$ are priced in nominal bonds differently from changes in $\mu_\pi(s_{1,t})$.

To what extent is this model reasonable to capture the dynamics of consumption growth and inflation? What are the implied responses of the variables to exogenous shocks? Are they similar to any results in the literature? To give initial answers to these questions I consider the Autocovariance Functions and the Impulse-Respond functions of the state-space.[10]

First, Figure 1 depicts the Autocovariance Functions of consumption growth and inflation data and the average Autocovariance Functions for these same variables implied by the model. Recall that the Autocovariance Function of a time series characterizes it. Except for some instances the data falls in the confidence intervals and the Autocovariance Function of the data is close to the average Autocovariance Function implied by the model, which provides support for the proposed model.[11]

Unconditionally, consumption growth is less persistent than inflation. There is a negative relationship between consumption growth and inflation, for as much as 5 quarters. These estimates are broadly comparable to those obtained, for example, in Walsh (2003), where the GDP deflactor is found to be negatively correlated with output for lags and leads, suggesting that fluctuations are mainly driven by supply shocks or by demand shocks with sticky prices.

Second, consider the Impulse-Response (IR) functions of the state-space. In order to identify the shocks impinging on the state-space I use the Cholesky decomposition on the estimated variance-covariance matrix. I assume that contemporaneously only an inflation shock affects consumption growth.

Figure 2 presents the Impulse-Response functions associated to the state-space. If an inflation shock affects consumption growth contemporaneously, a one standard deviation (0.29) surprise leads to an immediate 0.09 percentage points (i.e. 0.36% in annual terms) decrease in consumption growth. It has a cumulative effect of approximately 1% in annual terms for the following 5 quarters. A one standard deviation (0.53) surprise in consumption growth leads to a maximum increase in inflation of 0.09 percentage points (i.e. 0.36 % annual),

[10]For these Autocovariance Functions and the Impulse-Respond functions I consider the state-space with no regimes. I discuss the estimation below.

[11]Strictly speaking it does not give us evidence against the model.

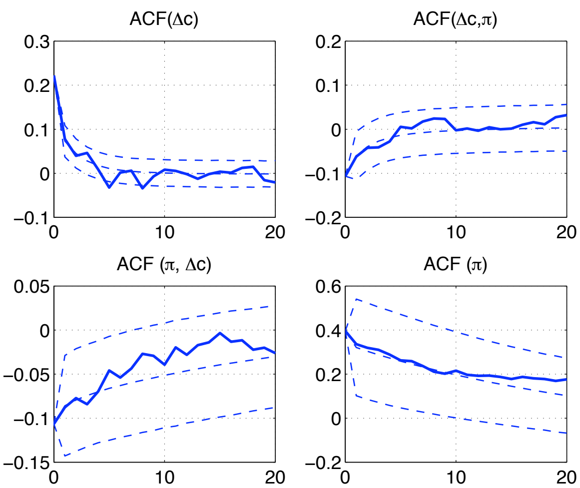

Figure 1: **Autocovariance Functions.**

The Autocovariance Functions for the observed consumption growth and inflation. Time unit is quarters in the x-axis. The confidence intervals (at 90% confidence level) and the average Autocovariance Function are constructed with bootstrapping.

yet the effect is not immediate. The inflation's response has no contemporaneous reaction, as mentioned. This can be understood as inflation having some stickiness. Money neutrality is observed in the long run. Centrally, inflation surprises bring information about future consumption growth prospects.

These results are statistical estimates and can be contrasted with some models in the literature.[12] Yet models describe the underlying dynamics of the variables, the direct estimates might have the influence of other variables that obscure these dynamics. Yet, for this model, the relationship between the fundamentals is taken as exogenous.

Finally, why is this model potentially useful to think about consumption growth, inflation dynamics *and* their relationship to yields? One reason is as follows. The Expectation Hypothesis (EH)[13] tends to hold at longer horizon

[12] A segmented markets model predicts a decrease in consumption to a money growth surprise if enough agents cannot access the financial markets immediately. In contrast, Lucas (1972) predicts a positive increase in production to a surprise in money growth. Note that the mapping of the variables used in each case is not direct.

[13] The EH can be defined on levels or on logarithms, the Jensen's inequality term makes them differ. Also, risk-neutrality is a necessary condition for the EH (defined on levels) but

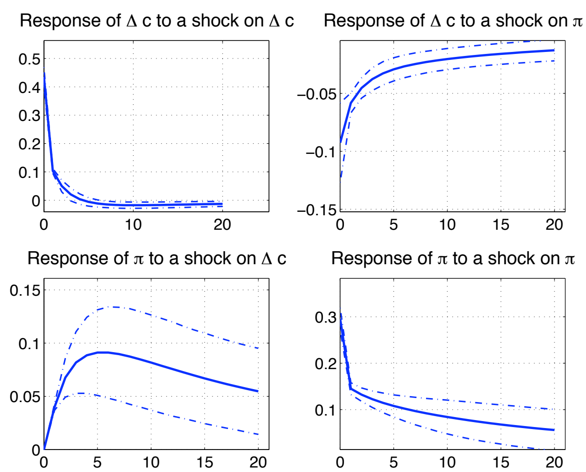

Figure 2: **Impulse-Response Functions.**

Impulse-Respond Functions to one standard deviation shocks. The identifying assumption is that only inflation affects consumption growth contemporaneously. The y-axis indicates percentage change, quarter base. For example, a one standard deviation π shock equals $1.16\ \% = (0.29 \times 4)$ in annual terms. Time units are quarters in the x-axis. The confidence intervals are constructed using Gibbs-sampling at 90%.

and on average.[14] If we fix $\mathbf{x_t}$, giving it its long-run mean value, i.e. $\mathbf{0}$, fix $\mu_\pi(s_{1,t})$, and fix the variance-covariance matrix, to obtain the yields, the EH actually holds in the model. So from the outset we know that on average it will hold.[15] Cochrane (2001), page 428, argues, the EH is a "slideshow." It is then explored whether the sluggishness of the variables, the structure of the regimes, and the recursive preferences can account for the predictability and, thus, the failure of the EH.

4 Preferences

Under Epstein-Zin (1991) and Weil (1990) preferences with an elasticity of intertemporal substitution ψ equal to one, the utility is given by the following recursive expression:

$$V_t = C_t^{1-\beta} \mathsf{CE}_t \left(V_{t+1}\right)^{\beta}, \tag{2}$$

where $\mathsf{CE}_t(V_{t+1}) = \mathbf{E}_t \left(V_{t+1}^{1-\gamma}\right)^{1/(1-\gamma)}$ is the certainty equivalent of the continuation value V_{t+1}. The coefficient of risk aversion is γ. The subjective discount factor β is positive and strictly less than one.[16] Epstein-Zin preferences have two distinctive features. First, the intertemporal elasticity of substitution ψ and the coefficient of relative risk aversion γ are not tied as in the Constant Relative Risk Aversion (CRRA) preferences case. Second, the Epstein-Zin preferences capture the intertemporal distribution of risk. How concern the agent is about the intertemporal distribution of risk is measured by γ.

Informally, I refer to the agent's concern for the intertemporal distribution of risk as a taste for persistence. Specifically, if $\gamma > 1$ the agent dislikes persistence. If $\gamma = 1$ the agent is indifferent to persistence. If $\gamma < 1$ the agent likes persistence. The formal definition is whether the agent prefers an early resolution of uncertainty vis a vis a late resolution of uncertainty.[17] For the

not a sufficient one.

[14] Paraphrasing Cochrane (2001), page 427.

[15] In fact, the EH holds for the model without regimes. We could, in this instance, observe deviations from it, depending on the behavior of $\mathbf{x}_t$. Yet in the model, it is the presence of regimes that is key to account for the deviations from the EH.

[16] This assumption differs from the model in Piazzesi and Schneider (2006). They estimate a β greater than one and consider a finite horizon.

[17] Another interpretation is to think of two forces acting in opposite directions. There is a distaste for persistence and there is also a taste for smooth consumption paths ruled by ψ, the elasticity of intertemporal substitution, which is 1 in this case. When γ=1 these effects cancel each other out. See Restoy and Weil (1998) for a similar interpretation. Another

most part, I use the taste for persistence interpretation. Estimates show that $\gamma > 1$, thus, the agent dislikes persistence, or equivalently, he prefers an early resolution of uncertainty.

In general, the logarithm of an uppercase variable is denoted by its lowercase, e.g. $\log V_t = v_t$. Equation (2) can then be written as:

$$v_t = (1-\beta)c_t + \beta(1-\gamma)^{-1}\log \mathbf{E}_t(\exp((1-\gamma)v_{t+1})). \tag{3}$$

This expression can be reinterpreted as the value function of an agent with a linear utility function having the risk-sensitivity operator for the continuation value given by $T(v) = (1-\gamma)^{-1}\log \mathbf{E}_t(\exp((1-\gamma)v))$. See Whittle (2002) for a general description. Tallarini (2000) uses it in a business cycles model. Also, it has a direct link to a problem with an agent that has specification doubts about his model, see Hansen and Sargent (2007). To solve for the value function we have the following proposition, which is a particular case of results in Hansen (2006) and Hansen (2007).

Proposition 1 The solution to value function in (3) under the state-space in (1) can be expressed as:

$$\begin{aligned} v_t &= c_t + \mathbf{E}_t \sum_{k=1}^{\infty} \beta^k \Delta c_{t+k} + f(s_{2,t}) \qquad (4) \\ &= c_t + \beta(1-\beta)^{-1}\mu_{\Delta c} + e_1'\beta(I-\beta\phi_x)^{-1}x_t + f(s_{2,t}) \end{aligned}$$

where $e_1 = (1\ \ 0)'$.[18]

Proof See Appendix A.

An increase in consumption or in expected discounted consumption growth, increases the continuation value. The function f captures the contribution of the changes in regimes states $s_{2,t}$ to the continuation value. f takes two values, i.e. the same number of values as the number of regime states. $f(s_{2,t})$ is a vector and $s_{2,t}$ is an indicator function pointing to the relevant entry.[19] There are two effects contributing to its value. First, an increase in the volatility (i.e. a regime state switch in $s_{2,t}$) decreases the continuation value. The agent dislikes risk. Second, the more persistent the regime $s_{2,t}$ is, the greater the difference

interpretation mentioned below is the change of measure, where the probabilities are tilted.

[18] If the regime $s_{2,t}$ does not exist $f(s_{2,t})$ is a constant.

[19] See Hansen (2006) for a more general exposition.

$f(s_{2,t}=2)-f(s_{2,t}=1)$. The agent dislikes persistence.[20]

5 Stochastic Discount Factor

In this section I obtain the Stochastic Discount Factor (SDF) and describe how it depends on tomorrow's state variables and regimes states differences, which can be interpreted as "revisions." The magnitudes of these "revisions" depend on the persistence of the variables and the regimes. This section draws from Hansen (2006), Hansen (2007), and Hansen, Heaton, and Li (2008).

The real SDF is given by

$$M_{t+1}^{(r)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \left(\frac{V_{t+1}}{\mathsf{CE}_t\left(V_{t+1}\right)}\right)^{(1-\gamma)}. \tag{5}$$

Noticing that

$$\mathbf{E}_t\left(\left(\frac{V_{t+1}}{\mathsf{CE}_t\left(V_{t+1}\right)}\right)^{(1-\gamma)}\right) = 1,$$

we can interpret (5) as a SDF under logarithmic preferences with a change of measure that depends on γ. This change of measure tilts the probabilities pessimistically when $\gamma > 1$. Thus, its presence allows for more variability in the SDF compared to Constant Relative Risk Aversion (CRRA) preferences.

By the solution to the continuation value (4) and some algebra the (log) SDF can be expressed as:

$$\log\beta - \Delta c_{t+1} - (\gamma-1)(\underbrace{(\mathbf{E}_{t+1}-\mathbf{E}_t)\sum_{i=0}^{\infty}\beta^i\Delta c_{t+1+i}}_{\text{Consumption growth "revisions"}} + \underbrace{f(s_{2,t+1})-\beta^{-1}f(s_{2,t})}_{\text{Regimes "revisions"}})$$

There are various effects over and above the CRRA preferences. As a standard result, as consumption growth Δc_{t+1} decreases, $m_{t+1}^{(r)}$ increases. Epstein-Zin utility introduces a preference for the temporal distribution of risk. This paper estimates γ to be greater than one, which means that the agent dislikes persistence, as mentioned. There are two components multiplied by $(\gamma-1)$: the consumption growth "revisions" and the regime "revisions." If there is no pre-

[20]Consider equation (4), the regime $s_{1,t}$ affecting the mean inflation does not appear in the continuation value. In contrast, the inflation component $x_{2,t}$ does affect the continuation value, it provides information on future consumption growth prospects. Note that v_t conditional on the information up to time $t-1$ and on the regime at time t has a normal distribution.

dictable component in consumption growth the first component is $\Delta c_{t+1} - \mu_{\Delta c}$. If there is less persistence in the regimes the second component is smaller. If Ω has no regimes, then the second component is constant.

Any changes in the "revisions" component $(\mathbf{E}_{t+1} - \mathbf{E}_t) \sum_i \beta^i \Delta c_{t+1+i}$ are a concern to the agent. For example, an upward revision in consumption growth decreases the stochastic discount factor, and vice versa. As estimates show, a positive revision in inflation signals a decrease in consumption growth.[21] This means that the inflation components $x_{2,t+1}$ and $x_{2,t}$ affect the real SDF; as explained, inflation has informational content for expected consumption growth. It follows that real bonds depend on the inflation components, perhaps initially a counterintuitive result.

As a consequence of the properties of f, an upward revision in the volatility regime (capture by $f(s_{2,t+1}) - \beta^{-1} f(s_{2,t})$) decreases the SDF, the agent dislikes risk. Moreover, persistence as measured by the transition probabilities $r_{i,i}$, will make the difference $f(s_{2,t+1}) - \beta^{-1} f(s_{2,t})$ larger, thus, decreasing the stochastic discount factor. The agent dislikes persistence. The agent knows that given a switch in regime state to a higher volatility state, and more persistence, higher volatility will be in the economy for a longer time. Intuitively, there is a "when it rains it pours feeling." These effects are exacerbated the bigger γ is.

The model for the SDF can be seen as a conditional linear factor of the form $m^{(r)}_{t+1} = \mathcal{A}(s_{2,t+1}, s_{2,t}) + \mathcal{B}'(\mathbf{x}_{t+1} - \mathbf{x}_t)$, where $\mathcal{A}$ and $\mathcal{B}$ are functions not defined here. This SDF might remind the reader to others in the literature, for example those in Ang et al. (2004) and Singleton et al. (2007). Theirs, however, although can be motivated as such, are not consumption-based models but directly rely on the existence of the SDF.

The nominal SDF is $M_{t+1} \equiv M^{(r)}_{t+1}/\Pi_{t+1}$, where Π_{t+1} stands for the price level at time $t+1$ *over* the price level at time t. It follows that the (log) nominal SDF is can be written as:

$$m_{t+1} = m^{(r)}_{t+1} - \pi_{t+1}$$

Thus, inflation has a double role in the (log) nominal SDF. The first one is within the consumption growth "revisions." The second is to standardized the real (log) SDF to have prices in nominal terms.[22]

[21] This relationship is explicit when the expectation operators are calculated.
[22] Another interpretation for the SDF is in the appendix.

6 Bond Pricing and Yields

An advantage of the model is that an exact solution for the prices of bond is possible. The derivations are based on results in Hansen, Heaton and Li (2008), Hansen (2006), and Hansen (2007). Let $P_t^{(n),r}$ denote the price of a real bond, i.e. a bond paying one unit of consumption, at time t and maturing in n periods. The price of a nominal bond, i.e. a bond paying a dollar, is denoted by $P_t^{(n)}$. The basic relationship for the price of a real bond is given by:

$$P_t^{(n),r} = \mathbf{E}_t\left(M_{t+1}^{(r)} P_{t+1}^{(n-1),r}\right). \tag{6}$$

Since $P_{t+n}^{(0),r} = 1$ and by the law of iterated expectations it follows that:

$$P_t^{(n),r} = \mathbf{E}_t\left(\exp\left(\sum_{k=1}^{n} m_{t+k}^{(r)}\right)\right) \tag{7}$$

where I have denoted $\log M_{t+k}^{(r)}$ by $m_{t+k}^{(r)}$. The price of a nominal bond at time t denoted by $P_t^{(n)}$, a bond that pays a dollar in n periods, can be written as:

$$P_t^{(n)} = \mathbf{E}_t\left(\exp\left(\sum_{k=1}^{n} m_{t+k}\right)\right) = \mathbf{E}_t\left(\exp\left(\sum_{k=1}^{n} m_{t+k}^{(r)} - \pi_{t+k}\right)\right), \tag{8}$$

where $M_{t+k} = M_{t+k}^{(r)}/\Pi_{t+k}$, Π_{t+k} is the price level at time $t+k$ *over* the price level at time $t+k-1$; so, $\log M_{t+k} = \log M_{t+k}^{(r)} - \log \Pi_{t+k}$. For a close solution we have the following results.[23]

Proposition 2. The price of a nominal bond at time t that matures in n periods conditional on the regimes states and on the state variables for the independent regimes case can be expressed as:

$$P_t^{(n)}(\mathbf{x}_t, \mathbf{s}_t) = \exp\left(-A(n) - \mathbf{B}(n)'\mathbf{x}_t - F(s_{2,t}, n) - G(s_{1,t}, n)\right),$$

where $A(n)$ is a scalar and $\mathbf{B}(n)$ is a 2×1 vector, $F(s_{2,t}, n), G(s_{1,t}, n)$ are functions that take the same number of values as regimes states, for a given n.

[23] In what follows I use the independent regimes case, yet analogous formulas hold for the general case (see Appendix C).

Explicitly,

$$\begin{aligned} A(n) &= -nh \\ \mathbf{B}(n) &= -\left(\sum_{i=0}^{n-1}(\phi_x')^i\right) w \text{ if } n>0 \text{ and } \mathbf{0} \text{ if } n=0. \end{aligned}$$

where $h = \log(\beta) + \mu_{\Delta c}$ and $w = -(1 \;\; 1)'$, these depend on the exact specification of the model.[24] F and G satisfy a set of recursive equations.

Proof See Appendix C.

For the price of the real bonds we have the following corollary.

Corollary 1 The price of a real bond at time t that matures in n periods as a function of the state variables and regime $s_{2,t}$ can be written as

$$P_t^{(n),r}(\mathbf{x}_t, s_{2,t}) = \exp\left(-A_r(n) - \mathbf{B}_r(n)'\mathbf{x}_t - F_r(s_{2,t}, n)\right),$$

where, as above, $A_r(n)$ is a scalar and $\mathbf{B}_r(n)$ is a 2×1 vector, $F_r(s_{2,t}, n)$ is a function that takes the same number of values as the regimes' states, for a fixed n. $A_r, \mathbf{B}_r(n)$ are:

$$\begin{aligned} A_r(n) &= -nh_r \\ \mathbf{B}_r(n) &= -\left(\sum_{i=0}^{n-1}(\phi_x')^i\right) w_r \text{ if } n>0 \text{ and } \mathbf{0} \text{ if } n=0. \end{aligned}$$

where $h_r = \log(\beta) + \mu_{\Delta c}$ and $w_r = -(1 \;\; 0)'$. Analogously, these depend on the exact specification of the model. F_r satisfies a set of recursive equations. Note that the regime associated to mean inflation drops out, a result reminiscent of the structure of the function f in the (log) continuation value v.

Proof See Appendix C.

The nominal yield of a bond is defined by $y_t^{(n)} \equiv -\log(P_t^{(n)})/n$, while the real yield as $y_t^{(n),r} \equiv -\log(P_t^{(n),r})/n$. Thus, the expression for the yields are obtained directly.

To finalize this section, I show some useful formulas that tie the nominal and real yields to the macroeconomic variables in most specifications of the model. The following relationships hold if we fix regime $s_{1,t}$, and we only have

[24] For example, the no regimes specification of the model does not have an explicit solution for A.

the regime, $s_{2,t}$, associated with the variance-covariance matrix.[25]

$$\begin{aligned} y_t^{(n)} - \mathbf{E}y_t^{(n)} &= \frac{1}{n}\mathbf{E}_t \sum_{i=1}^{n}(\Delta c_{t+i} - \mathbf{E}\Delta c_{t+i}) + \frac{1}{n}\mathbf{E}_t \sum_{i=1}^{n}(\pi_{t+i} - \mathbf{E}\pi_{t+i}) + \\ &\qquad + (F(s_{2,t}, n) - \mathbf{E}F(s_{2,t}, n))/n \\ y_t^{(n),r} - \mathbf{E}y_t^{(n),r} &= \frac{1}{n}\mathbf{E}_t \sum_{i=1}^{n}(\Delta c_{t+i} - \mathbf{E}\Delta c_{t+i}) + (F_r(s_{2,t}, n) - \mathbf{E}F_r(s_{2,t}, n))/n. \end{aligned}$$

The no regimes and regime cases of these formulas have an important difference. While the versions of these equations in the case of no regimes are independent of the value of γ, the cases with regimes are affected by γ through F and F_r. More general relationships in the case where there is a regime in $\mu_\pi(s_{1,t})$ are not possible with the nominal yields. To the best of my knowledge, an explicit solution for the components of the yields capturing the change in $s_{1,t}$ is not attainable.

7 Estimation

The model is estimated in two steps. First, the state-space is estimated with Maximum Likelihood (ML) or Gibbs-sampling (GS) methods. Second, the preference parameters, γ and β, are obtained, minimizing the distance between the cross-sectional average of the yields implied by the model and the yields in the data.

There are at least two reasons for using a two-steps procedure.[26] First, it simplifies the estimation. The likelihood function of the state-space with no regimes is already challenging to estimate. Thus, adding regimes adds to this challenge. Second, there is a need to separate the macroeconomic from the financial variables, since the latter are better measured.[27]

The state variables $\mathbf{x}_t$ are estimated assuming $\mathbf{x}_0 = \mathbf{0}$ and using the state-space to back them out.[28] For the probability of being in a regime state consider

[25] They are a generalization of the case for no regimes as in Piazzesi and Schneider (2006).

[26] A joint estimation is an alternative that was only initially pursued. The key step is to model the measurement errors of the yields explicitly. Assume these measurement errors are independent to the shocks impinging the state-state and, finally, construct the likelihood to use ML. Details are given in Appendix E.

[27] Financial variables are not exempt from measurement problems, e.g. microstructure phenomena.

[28] The results are robust to the value of the initial point, $\mathbf{x}_0$. The second possibility in under a joint estimation by assuming there are two (i.e. the same number as state variables) yields measured without errors. Then, the $\mathbf{x}_t$ variables are backed out from these yields.

the following. Recall the notation $\mathbf{z}_{t+1} \equiv (\Delta c_{t+1} \quad \pi_{t+1})$ and denote $\mathbf{z}^{(t)}$ as $\{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_t\}$, which collects the history up to time t of variable $\mathbf{z}$. To have a probability assessment of the current regime either, $Pr(s_t|\mathbf{z}^{t-1})$, the optimal inference or, $Pr(s_t|\mathbf{z}^T)$, the optimal smoothing, can be used. I use the optimal inferences. For the pricing, to determine which one is the prevalent regime state I choose the regime state with the highest probability at time t. Estimating $\mu_\pi(s_{1,t})$ is problematic, given that $x_{2,t}$ introduces noise. For the estimations, I fix specific estimates on plausible values for $\mu_\pi(s_{1,t})$.[29] This amounts to using GS with an informative prior.

Regarding identification we have the following issues. Given that ϵ is normally distributed the likelihood function is relatively straightforward to obtain. The Gibbs-sampling algorithm is not difficult to construct, given the availability of the conditional density functions (except for K). As an identification assumption I set $\mathbf{x}_0 = [0 \quad 0]'$, as mentioned. Under these assumptions $\Omega(s_{2,t})$, $\mu_{\Delta c}, \mu_\pi(s_{1,t})$, ϕ_x, K, Q, and R are identified. An estimate of $\mathbf{x}_t$ can be recovered as described above. See Appendix E for further details.

I estimate the following specifications of the model:

1. **N̲o regimes:** There are no regimes affecting the state-space.

2. **H̲eteroscedasticity:** regime $s_{1,t}$ is fixed, i.e. μ_π is constant, and regime $s_{2,t}$ affects $\Omega(s_{2,t})$.

3. **I̲ndependent:** regime $s_{1,t}$ affects $\mu_\pi(s_{1,t})$, and $s_{2,t}$ affects $\Omega(s_{2,t})$, and regimes $s_{1,t}$ and $s_{2,t}$ are independent.

4. **S̲ame:** $s_{1,t}$ affects $\mu_\pi(s_{1,t})$ and $\Omega(s_{1,t})$. In other words, $s_{1,t} = s_{2,t}$.

Whether consumption growth has heteroscedastic shocks, in particular in the last decades, is a matter of debate.[30] For all the specifications I assume that $\Omega(s_2 = 1)_{11} = \Omega(s_2 = 2)_{11}$. Regimes in Ω_{22} are motivated by the observation that inflation variance is time-varying. Whether the covariance is subject to regimes depends on how its regime structure relates to the variance of consumption growth and inflation.[31]

[29]The means estimated in specification **S** are used for specification **I**. Other criteria might be used, but at the same time ad-hoc choices for the means, $\mu_\pi(s_{1,t})$, should be avoided.

[30]This assumption differs in related papers in the literature, e.g. Bansal and Yaron (1995) assume it does. While, for example, Cochrane (2005) criticizes it.

[31]A possible way to think about it, is to decompose the shocks as functions of factors, say, $\epsilon_1 = \sum a_i Z_i$ and $\epsilon_2 = \sum b_i Z_i$, where the Z's are standardized normal i.i.d.. The regimes configuration then depends on how the regimes affect the coefficients a's and b's.

Given the structure of the regimes, in the estimates the regime with high variance is associated with high covariance, and vice versa. In this regime state, inflation shocks are on average bigger and they predict a much lower consumption growth, given the greater (in absolute value) negative covariance. Thus, a switch in the regime state affecting the covariance of inflation and consumption growth can be interpreted as a change in the price of risk.

Sims and Zha (2006) attempt to determine whether there were monetary policy regimes switches using a reduced form model. They conclude that the regimes are mostly associated with changes in the variance-covariance shock of their system. We could think of specification **H** as the closest to theirs. The regimes in $\mu_\pi(s_{1,t})$ are motivated by the drastic changes in inflation mean during the 1970s and 1980s. An assumption that is not obvious is the relationship between the regimes switches, $s_{1,t}$ and $s_{2,t}$. Although a general relationship is desirable, its estimation proved difficult.[32] This is reminiscent of the challenges estimating cross-correlations elsewhere in the model.

8 Data and Code

The consumption and price indices data are taken from the National Income and Product Accounts (NIPA). Only non-durable goods and services are considered to construct the consumption variable.[33] A corresponding price index is constructed as a measure of inflation.[34] The bond yields with maturities greater than one are from the Center for Research in Security Prices (CRSP) Fama-Bliss discount bond files. The short rate is from the CRSP Fama risk-free rate. The maturities are 1 quarter, 1 year, 2, 3, 4 and 5 years. A constant population growth is assumed, see Appendix E for details. Table 1 presents the basic statistics of the data used and Appendix D contains complementary information on these series.

[32]The specification for the general relationship between $s_{1,t}$ and $s_{2,t}$ was estimated. The model can be solved under this specification as well. Yet, it is unclear if a maximum is achieved in the optimization, thus, this case is not pursued. Also, the assumption of independence depends to an extent on the number of regimes states.

[33]This is a common procedure in the literature. The point is that we derive utility out of consuming a "dividend" of a durable good, not out of durable good itself. An additional model would have to introduced to account for this issue. The underlying assumption is that consumption for durable goods and services is separable.

[34]Its construction follows the one in Piazzesi and Schneider (2006). The construction of a corresponding price index for the consumption basket used is relevant. Some models in the literature, see e.g. Wachter (2006), uses the CPI directly although the consumption variable used is not a general consumption basket.

Table 1: **Basic Statistics.**

These are the basic statistics for consumption growth, inflation, and the yields used in the model. The time unit is quarters. For example, the annual average inflation is 3.71% = (0.9267×4), while the annual average 1 quarter yield is 5.15% = (1.2869×4). The series go from 1952:2 to 2005:3. The consumption is from NIPA data on non-durable goods and services. The inflation index is constructed from the corresponding NIPA price indices. The yields are from CRSP Fama-Bliss and CRSP Fama risk-free rate. m. stands for mean, s.d. for standard deviation, and ACC for autocorrelation coefficient with one lagged correlation.

	Δc	π	1Q	1Y	2Y	3Y	4Y	5Y
m	0.8230	0.9267	1.2869	1.3893	1.4408	1.4832	1.5147	1.5351
s.d.	0.5712	0.6293	0.7288	0.7307	0.7208	0.7029	0.6961	0.6849
ACC	0.3528	0.8471	0.9382	0.9468	0.9560	0.9620	0.9664	0.9695

From Table 1, inflation has a greater mean, has more variability, and is more persistent than consumption growth. The average mean of yields increases with maturity. Except for the maturity of 1 Quarter, volatility in yields decreases as maturity increases. Yields are more persistent than inflation and consumption growth, and the autocorrelation in yields increases slightly with maturity.

Table 2: **Correlation coefficients.**

This table presents the correlation coefficients of the main variables: consumption growth, inflation, 1 quarter, 1 year, ..., and 5 years yields.

	Δc	π	1Q	1Y	2Y	3Y	4Y	5Y
Δc	1.0000	-0.3595	-0.1790	-0.1600	-0.1522	-0.1544	-0.1601	-0.1593
π	-0.3595	1.0000	0.6790	0.6650	0.6408	0.6208	0.6082	0.6040
1Q	-0.1790	0.6790	1.0000	0.9884	0.9743	0.9598	0.9468	0.9360
1Y	-0.1600	0.6650	0.9884	1.0000	0.9931	0.9831	0.9722	0.9633
2Y	-0.1522	0.6408	0.9743	0.9931	1.0000	0.9970	0.9914	0.9860
3Y	-0.1544	0.6208	0.9598	0.9831	0.9970	1.0000	0.9980	0.9950
4Y	-0.1601	0.6082	0.9468	0.9722	0.9914	0.9980	1.0000	0.9985
5Y	-0.1593	0.6040	0.9360	0.9633	0.9860	0.9950	0.9985	1.0000

From Table 2, inflation correlates negatively with consumption growth; thus, these estimations are consistent with the idea that high inflation is associated with low consumption growth. Yields of different maturities are highly correlated, which diminishes as the maturity difference increases. Inflation maintains a positive correlation with all the yields, while consumption growth has a negative correlation with all the yields. A key goal is to understand the interaction between these variables and how it might have changed.

Most algorithms are coded and implemented by the author.[35] The algorithms were tested by generating simulated data for which the parameters were known, under these circumstances they provide reasonable estimates. The price index construction script is taken from Piazzesi and Schneider (2006).

9 Maximum Likelihood Estimation

For the Maximum Likelihood Estimation (MLE) under regime switching I follow Hamilton (1994).[36] The probability density function conditional on the regime state is: $f(\mathbf{z}_{t+1}|s_t)$. Given that the regimes states are latent, an observation could have come from any of the regime states. The optimal inferences are used, $Pr(\mathbf{s}_t|\mathbf{z}^{(t)})$, to assess the probability of the prevalent regime state, as mentioned (see Appendix E for details).

Table 9, in the appendix, presents the estimates for the case of $s_{1,t}$ being fixed and a regime switch in $\Omega(s_{2,t})$, specification **H**. Table 10 presents the estimates for the case of independent regimes $s_{1,t}$ and $s_{2,t}$, associated with mean inflation and the variance-covariance matrix, respectively, i.e. specification **I**. Table E presents the estimates for the case in which $s_{1,t} = s_{2,t}$, specification **S**.

The estimates for $\Omega(s_{2,t})$ do not overlap the confidence interval for any of the parameters of the other regime state, evidence for the presence of regimes. The negative conditional covariance is present in both regimes states. One regime state is associated with high volatility in which shocks to inflation and their effects on consumption are bigger (in absolute value).

The persistence of $x_{1,t}$ and $x_{2,t}$ can be measured with the eigenvalues of ϕ_x or with the half-lives of the shocks on ϵ, as mentioned. Consider Tables 8, 9, and 10 in Appendix E. The inflation component is much more persistent. The consumption growth component is slightly persistent. The variance of the consumption growth component, $e_1' K \Omega(s_{2,t}) K' e_1$, is small relative to the shocks affecting consumption growth, $e_1' \Omega(s_{2,t}) e_1$. The persistence of the

[35]They were coded and implemented in MATLAB Version 7.8.0.347 (R2009a). The toolboxes Statistics and Optimization are needed to run the codes.

[36]Hamilton (1994) offers two algorithms: the direct optimization of the Maximum Likelihood and the E.M. algorithm, for convenience I use the former.

regimes is measured by the transition probabilities in the matrices associated to the regimes, Q and/or R or with the expected time in a regime state, given, e.g. by $1/(1 - r_{i,i})$.

The parameters that are not affected by the regimes are close to the estimations in other specifications, e.g. see Table 8 in Appendix E. However, an important difference between the specifications is the estimate of $\mathbf{x}_t$. The exact configuration of the regimes does have implications on this object.

Regime Tests

The presence of regimes switching has economic motivations. Yet, it is nevertheless useful to run some statistical tests. To test for the presence of regimes or the number of regime states is challenging.[37] The common tests, e.g. the likelihood ratio, do not apply since under the null hypothesis a set of parameters is generally unidentified. To overcome this difficulty, I use a Modified Likelihood Ratio test for regime switching (MLR) by Kasahara et al.(2008).[38] The statistic is $MLR \equiv \max_p \max_\theta \left(L(\theta, p) - L(\theta_{Rest}, p)\right)$, which under the null has as a distribution $(\max\{G, 0\})^2$, where G has a normal standardized distribution. L is the likelihood function using the unconditional probabilities of being in a regime plus a penalty function of the form $Tc_T \log(p(1-p))$, where p is the unconditional probability of being in a regime state, and c_T converges to 0 as T tends to infinity. This penalty function avoids the unidentification of p under the null (i.e. no or less regimes). The distribution of MLR can be tabulated, the critical values are 2.8 at 10% and 3.8 at 5%. Where applicable, I use the usual Likelihood Ratio test, $-2(L(\theta_{Rest}) - L(\theta_{UnRest}))$. Table 3 presents the statistics. Overall, statistically, the case for regimes is supported, see the first row in the table.[39] Loosely speaking, the test comparing specifications **S** and **H**, supports the former. The tests comparing **I** to either **S** or **H**, support **I**.

[37] The presence of regimes refers to comparing one against more regimes and the number of regime states to comparing two against more.

[38] Refer to Kashara et al.(2008) for details.

[39] A third regime state is a possibility that was not pursued in any of the regimes switches.

Table 3: **The Modified and Common Likelihood Ratio Tests.** The * indicates that the test failed to reject the null at 10%. The letters, **N**, **H**, **I**, **S**, stand for No regimes, Heteroscedasticity (regime just in Ω), Independent ($s_{1,t}$ and $s_{2,t}$ are independent), and Same ($s_{1,t} = s_{2,t}$), respectively. The null is the specification that restricts, i.e. no regimes or less regimes. A † indicates it is a common likelihood ratio test, otherwise it is a modified likelihood test. The critical values for the MLR are approximately 3.85 at 5% and 2.69 at 10%. The specification of the row is the null hypothesis, i.e. the one that restricts the estimation compared to the column.

Specification	**N**	**H**	**I**	**S**
N	-	5.11	5.58	5.40
H	-	-	81.1	13.45†
I	-	-	-	-
S	-	-	26.29†	-

10 Gibbs-Sampling Estimation

The Gibbs-sampling (GS) estimation method[40] provides several tools for the present analysis: a) the estimates of the parameters as a basic output; b) the posterior distributions of each parameter to construct the confidence intervals for the Impulse-Respond Functions (see Figure 2); c) the posterior distribution of the eigenvalues of the matrix ϕ_x, to assess the variables' stationarity and the validity of taking the limit of summations of the form $\sum_i^n \beta^i \phi_x^i$; d) the estimates of a set of statistical tests to assess the persistence of the component in consumption growth; and, e) the posterior distributions of the short-run, long-run responses, and infinite discounted summations to shocks. Except for the second one. These are examined below.

The pseudo-code of the algorithm used is in Appendix E. The priors used for GS parameters are uninformative, except for the case of the K which is passed from the MLE estimate. Figures 3 and 4 presents the posterior estimates distribution for key parameters of the state-space, ϕ_x and Ω, in the no regimes case.

The posterior distributions of the elements $\phi_x[i, j]$ where $i \neq j$ show the difficulty in assessing the economic and statistical significance of the cross ef-

[40] Gibbs-sampling provides approximate samples of a distribution when direct sampling is difficult and access to the marginal distributions is possible. It has parameter uncertainty built into. It was proposed by Geman and Geman (1984) and is a special case of the Metropolis-Hastings algorithm (see Appendix E).

fects, both in the short and long run. This is particularly the case for $\phi_x[1,2]$. Although in the tails, the support of both posterior distributions contain the zero. In contrast, the elements in $\phi_x[i,j]$ where $i = j$, convey the significance of the direct effects of the variables. A crucial point is the extent to which these objects and their measurements, relating the short run dynamics of the variables, are adequate to evaluate the long run effects. As for the posterior distributions of Ω, the economic and, statistical, conditional covariances are less of an issue, compared to the non-diagonal elements in ϕ_x.

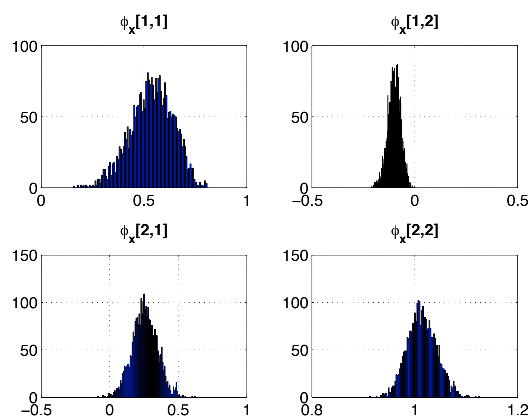

Figure 3: **Posterior distribution of ϕ_x.**

The priors used are uninformative except K which is passed from MLE. The notation $[i,j]$ denotes the i^{th} and j^{th} entries.

The Variance Ratio test

A central assumption in the model is that consumption growth is not a random walk. In contrast, at least since Hall (1978), the assumption that consumption growth follows a random walk assumption has been used in the literature.[41] These competing hypotheses for the distribution of consumption growth are central to the use of Epstein-Zin preferences. When consumption growth is a random walk, Epstein Zin preferences and CRRA preferences are observational

[41] Two related issues are: i) the exact components used consumption, e.g. whether it has durables goods or not; and, ii) the information set used, since other variables might provide information that might influence the estimation.

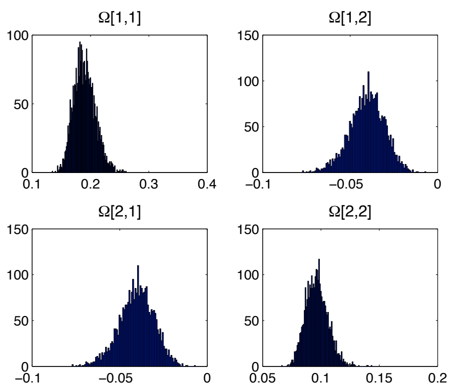

Figure 4: **Posterior distribution of Ω.**

The priors used are uninformative except K which is passed from MLE. The notation $[i, j]$ denotes the i^{th} and j^{th} entries.

equivalent[42] (Kocherlakota (1990)). The Variance Ratio is $\frac{1}{K}\frac{Var(c_{t+K}-c_t)}{Var(c_{t+1}-c_t)}$, used in Cochrane (1988). Under the null of a random walk its distribution has mean 1, for all $K = 1, 2, 3, ...$ I perform simulations using Gibbs-sampling to obtain the distribution of the Variance Ratio under the observed and under the null. Figure 5 presents the distribution for three lags, 6 months, 1 year and 2 years ($K = 2, 4$ and 8) under the data. The estimated statistics are $1.3417, 1.7731$ and 2.0773, respectively. All of them fall outside the support of the simulated distribution under the null. Thus, this provides evidence against the random walks hypothesis. A component such as $x_{1,t}$ would explain deviations from the null in the observed direction. The question is what explains $x_{t,1}$ economically, beyond a statistical interpretation.

The information set used is central for predictability. The variables in the state-space are the only ones assumed to be in the information set of the agent. They consequently define ex-ante the macroeconomic risk in the model, yet others might be relevant. Some papers incorporate other macroeconomic variables, e.g. Hansen, Heaton and Li (2008) use corporate earnings, into their asset pricing model. In our case, the use of a third variable in the model is to have information beyond and above to that conveyed by inflation.[43]

[42]Hansen and Sargent (2006) take a step further and consider a model in which the agent entertains the possibility of both models for consumption growth, and has specification doubts on his model.

[43]I explored this with initial estimates. The appendix has estimates using as a third variable

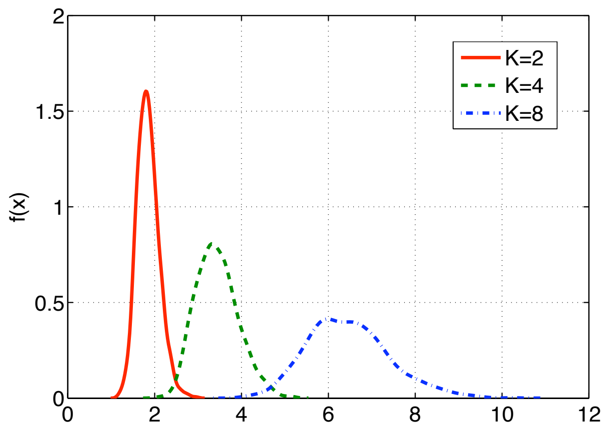

Figure 5: **Posterior distribution of the VR Statistic.**

The posterior probability density function is presented for $K = 2, 4$ and 8, standing for half a year, one year and two years.

Posterior distributions of key elements

The model measures elements in the long run; thus, it is important to examine the statistical precision of their measurements. I consider the posterior distributions of six elements. First, the response of consumption growth to a shock to inflation, after one quarter and after 20 quarters. They measure the precision of the information content in inflation for future consumption growth prospects. Second, the distribution of the four elements in $\sum_i^\infty \beta^i \phi_x^i = (I - \beta\phi_x)^{-1}$, reflect the discounted expected responds of consumption growth and inflation. Note that the convergence is influenced by β.[44] It is similar to the expression $\sum_i^n \phi_x^i$, used in the yields (recall the definition of $\mathbf{B}(n)$). Thus, the analysis should shed light on it as well. The estimate, ϕ_x^n, is key to the implications of the model. It relates to the measurement of the prices of risk, and the expected behavior of variables in the near and far future.

Figure 6 suggests that unexpected positive inflation brings about negative

labor income growth. It is not clear whether it has information not contained in consumption growth and inflation.

[44]This object is reminiscent to the Gordon formula, which is used to price stocks, $P = \frac{D}{g-r}$ where P is the price of the stock, D is the dividend of the share, g the rate the dividend grows each period, and r is the rate of discount of the cashflow. Small changes to either the discount rate or dividend growth have important implications on the price.

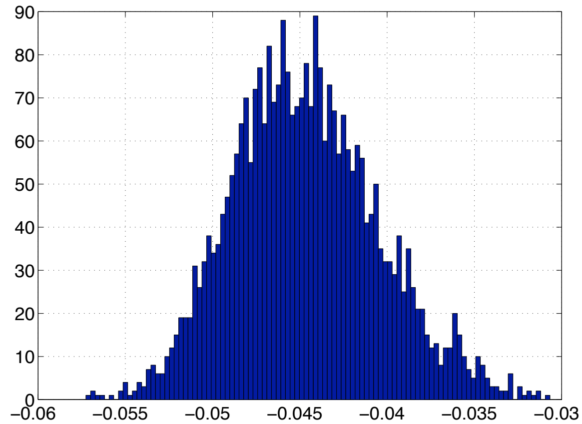

Figure 6: **Short run respond of Δc to a shock to π.**

Posterior distribution of a short run respond of Δc to a shock to π. The units are in quarter percentages. The shock has the standard deviation of conditional inflation 0.29, the short run is understood as 1 quarter.

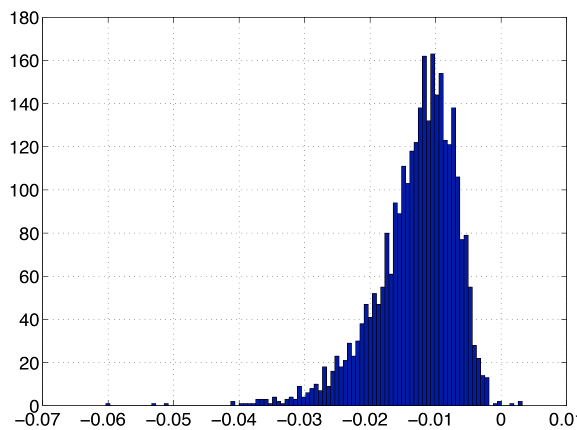

Figure 7: **Long run respond of Δc to a shock to π.**

Posterior distribution of a long run respond of Δc to a shock to π. The units are in quarter percentages. The shock has the average size standard deviation of conditional inflation, 0.29, long run is understood as 20 quarters.

consumption growth prospects. A shock of 0.29, signals a 0.04 decrease, 0.16% in annual terms. Considering the long run respond, in Figure 7, we see that the evidence is not as strong regarding the effects of inflation on consumption growth in the long run. The posterior distribution of the respond has zero in its support and is negatively skewed. The mean respond is around 0.06% in annual terms. It is uncertain whether the measurement of consumption growth has enough precision to make this respond significant. Parameter uncertainty makes the posterior distribution for the long run shock to have a similar variance to that of the short run shock. This happens although we would expect that the economic effects decrease through time.

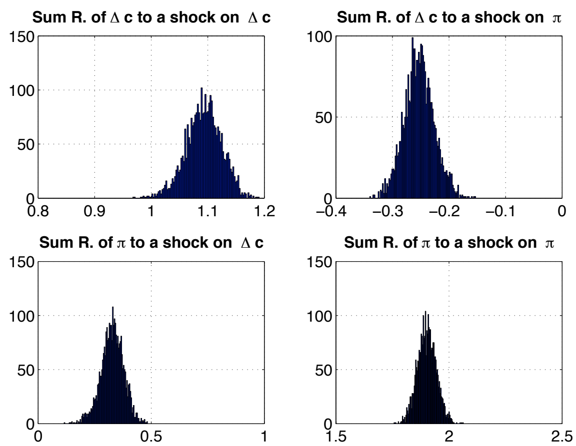

Figure 8: **Respond of $(I - \beta\phi_x)^{-1}$ to shocks to Δc and π.**

Posterior distribution of the respond of $(I - \beta\phi_x)^{-1}$ to shocks to Δc and π with β=0.995. The units are in quarter percentages.

Figure 8 presents the four posterior distributions for the elements in $(I - \beta\phi_x)^{-1}$. The plots allow us to be quite confident regarding the responds and expected values of the variables to shocks on each one. The measurement of the cross-effects is more problematic. A consumption growth summation respond is approximately -1% in annual terms. While an inflation growth summation respond is approximately 1.2% in annual terms.

In sum, the measurements of elements, in particular the long run, are a cause for concern regarding their precision. Key posterior distributions are assessed to investigate the implications for the model. In the case of the long run responses, the measurement of ϕ_x has relevant implications and should be taken with care, specially the non-diagonal elements. In the case of the summation, the model relies on the effects building up.

11 Implied Yields

We are now in a position to analyze the implied yields. Thus, consider each of the components that are part of the nominal yield formula.

$$y(\mathbf{x}_t, s_t) = A(n)/n + B(n)'\mathbf{x}_t/n + F(n, s_{2,t})/n + G(n, s_{2,t})/n$$

First, the component $A(n)/n = \log(\beta) + \mu_{\Delta c}$ is constant through the maturities.[45] A higher β, a more impatient agent, delivers a higher level of $A(n)/n$. As he wants to borrow more, the yields increase to make it more expensive to do so and, thus, reach an equilibrium. Similarly, a higher $\mu_{\Delta c}$, structurally a more vibrant economy, gives a higher $A(n)/n$. As the agent insists on borrowing to smooth his consumption, yields need to increase to reach an equilibrium. For example, with a estimate of $\beta = 0.995$ and $\mu_{\Delta c} = 0.8230$, I obtain a value of 2.42% for this components.[46]

Second, Figure 9 depicts the two components in $\mathbf{B}(n)'\mathbf{x}_t/n$, the component associated to consumption growth, $\mathbf{B}(n)_1 x_{1,t}/n$, and the component associated to inflation, $\mathbf{B}(n)_2 x_{2,t}/n$, as functions of time to maturity n. Since the mean value of $\mathbf{x}_t$ is zero, I depict these components with the standard deviations of $x_{1,t}/n$ and $x_{2,t}/n$. Most of the variation in nominal yields comes from inflation. Short term maturities are affected more than long term maturities for both components.[47] Since the variables in $\mathbf{x}_t$ are negatively correlated, we would typically see these lines on the opposite sides of the x-axis.

Two comments are in line, first, the persistence in the macroeconomic variables plays a double role. On the one hand, it defines the cross-sectional weights of the yields (through $\mathbf{B}(n)$). On the other, it defines the times series behavior of a shock on either variable in $\mathbf{x}_t$; thus, linking the cross-sectional and time series behavior of the yields. Second, $\mathbf{B}(n)/n$ does not depend on γ.

Third, consider Figure 10 showing the component $F(s_{2,t}, n)/n$, associated

[45] In the case of no regimes there in no explicit solution.

[46] $2.42 = 100(\log(0.995) + 0.00823)$ quarterly.

[47] Recall that $\mathbf{B}(n)/n = \left(\sum_{i=0}^{n-1} (\phi_x')^i\right) (1 \;\; 1)'/n$.

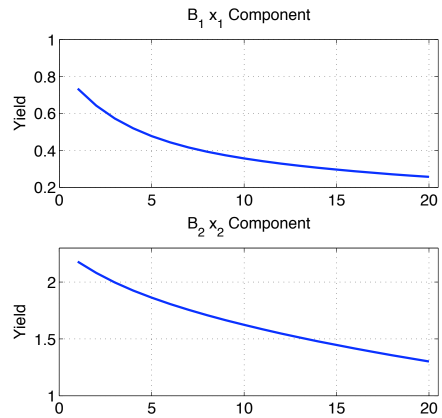

Figure 9: **Components of $\mathbf{B'x_t}/n$.**

The plots depict the contribution of the components $B_1(n)x_{1,t}/n$ and $B_2(n)x_{2,t}/n$ that forms the yield of a bond. The elements in $\mathbf{x}_t$ are assigned their standard deviation estimate. The x-axis indicates time to maturity. Time units are in quarters.

to the regime affecting the variance-covariance matrix. Its influence increases as maturity increases, in both regimes states. Thus, the presence of regimes contributes to the average positive slope of the nominal yield curve. Specifically, the figure depicts the function $F(n, s_{2,t})/n$ for different levels of $\alpha\Omega$ where α is a scalar with values $0.8, 0.9$ and 1.0.

There are three effects to consider. First, higher "risk" (i.e. α) implies a lower real component of nominal yields, a precautionary savings effect. Second, higher conditional inflation variance implies an increase in the nominal yields, as a compensation for inflationary risk. As risk increases the yields go down, i.e. the first effect dominates. Under this scenario, unexpected inflation shocks tend to be bigger and when they occur, they imply a bigger decrease in consumption growth. The regime state associated to a high conditional variance to inflation has a high conditional covariance of the shocks. In this case, the second effect dominates.[48] These effects assume no changes in the other components.

Figure 11 presents the component $F(s_{2,t}, n)/n$ for different levels of persistence in the regime. For simplicity, it is assumed that $r_{1,1} = r_{2,2}$. It illustrates that the more persistent is the regime associated to the variance-covariance

[48]Recall, $\Omega(s_1 = 1)_{1,1} = \Omega(s_2 = 2)_{1,1}$

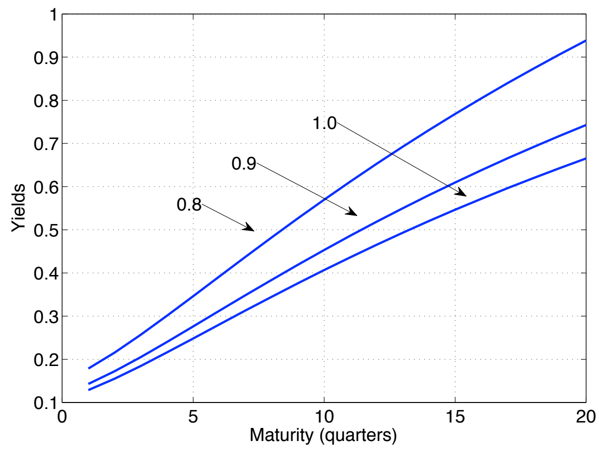

Figure 10: **Function $F(n, s_{2,t} = 1)/n$ for different levels of $\alpha\Omega$.**

The plot shows the function $F(n, s_{2,t})/n$ for different levels of $\alpha\Omega$ where α is a scalar with values $0.8, 0.9$ and 1.0. As risk (i.e. α) increases the agent steps on precautionary savings sending the yields down. The regime state $s_{2,t}$ is fixed.

matrix, the bigger the difference $(F(s_{2,t} = 2, n) - F(s_{2,t} = 1, n))/n$, and thus the changes in yields. In terms of regime switching, this is where much of the dynamics take place.

Finally, for $G(s_{1,t}, n)/n$, the component associated to the regime affecting the mean inflation, changes in inflationary regimes affect the short term and long term similarly. A regime switch to high inflation mean moves the yield curve up, for the most part, in a parallel fashion but affecting the short end of the curve the most, and vice versa. Analogously, the levels of $G(s_{1,t}, n)/n$ depend on the estimates of $\mu_\pi(s_{1,t})$ and on the persistence in the regime associated to μ_π (measured through Q).

12 Cross-sectional analysis

Consider Table 4, a positive slope on the average nominal yield curve is obtained for all the specifications. Most of the average yields are comfortably within a standard error of the data. The volatility for the implied yields is below and slopes downwards more rapidly compared to the data. Yet the presence of

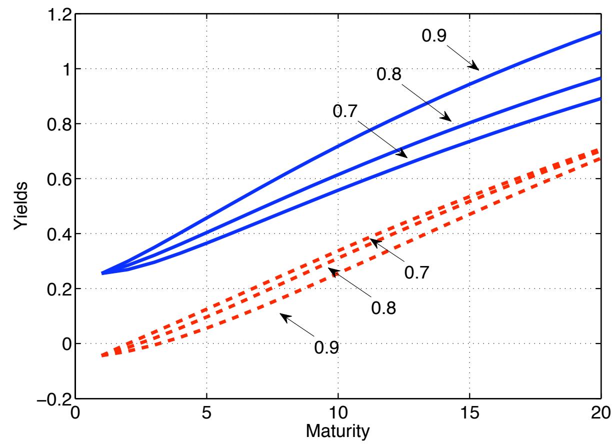

Figure 11: **Function $F(n, s_{2,t})/n$ for different levels of persistence.**

The plot shows the function $F(n, s_{2,t})/n$ for different levels of persistence of $s_{2,t}$ as measured by $r_{1,1}$ and $r_{2,2}$. The tags in the plot indicate their values. As the regime becomes less persistence $(F(n, 2) - F(n, 1))/n$ becomes smaller. F then depends on the value of the estimate at a regime state and its persistence. For simplicity the assumption $r_{1,1} = r_{2,2}$ is made.

regimes improves the variability at the long end of the curve.

The preference parameters are smaller to those reported in Piazzesi and Schneider (2006), with a similar model yet without regimes. An intuitive reason for this difference is that the presence of regime switching introduces more variation in the Stochastic Discount Factor (SDF). There is then less need to have a bigger coefficient γ. Thus, the presence of regimes not only allows to capture the inflation and consumption growth dynamics in a better way -by construction-, but also allows to obtain more reasonable estimates for the preference parameters. Nevertheless, it does not resolve the problem of obtaining what would be considered a reasonable magnitude of γ.[49]

Table 4 reports the implied real yields, for which the curve is downward sloping, consistent with the available evidence. It is not straightforward to compare these results against the data due to the short period Treasury Inflation-Protected Securities (TIPS)[50] bonds have been trading. The magnitudes and

[49]This problem directly relates to the Equity Premium Puzzle.
[50]Also known as index-linked bonds.

Table 4: **Cross-sectional means for nominal and real yields.** The specifications are: No regimes, Heteroscesdastic, Independent and Same regime. The preference parameters are on the last two columns. s.e. is standard error and sd() is standard deviation. *For the case of No regimes, the estimates of the preference parameters are hard to interpret because an adequate calibration is not achieved. The prime in the **I**' indicates that the means $\mu_\pi(s_{1,t})$ are fixed before the optimization.

		1Q	4Q	8Q	12Q	16Q	20Q	γ	β
Data	$\mathbf{E}(y)$	5.1477	5.5571	5.7633	5.9326	6.0588	6.1405	-	-
	s. e.	0.1988	0.1993	0.1966	0.1917	0.1899	0.1868		
	sd(y)	2.9152	2.9227	2.8832	2.8115	2.7843	2.7394		
	$\mathbf{E}(y^{(r)})$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.		
N	$\mathbf{E}(y)$	6.1502	6.3327	6.5729	6.7912	6.9848	7.1557	60*	0.9999*
	s.e.	0.1237	0.1126	0.1012	0.0918	0.0837	0.0767		
	sd(y)	1.8135	1.6509	1.4839	1.3461	1.2277	1.1245		
	$\mathbf{E}(y^{(r)})$	3.1522	2.9473	2.7899	2.6834	2.6004	2.5315		
H	$\mathbf{E}(y)$	5.4624	5.5613	5.7242	5.8900	6.0479	6.1949	40	0.9996
	s.e.	0.1257	0.1162	0.1074	0.1000	0.0934	0.0874		
	sd(y)	1.8430	1.7044	1.5748	1.4669	1.3700	1.2810		
	$\mathbf{E}(y^{(r)})$	2.6728	2.5186	2.3959	2.3086	2.2375	2.1763		
I'	$\mathbf{E}(y)$	5.6059	5.5699	5.7160	5.8810	6.0425	6.1948	48	0.9997
	s.e.	0.1313	0.1044	0.0962	0.0904	0.0852	0.0803		
	sd(y)	1.9247	1.5307	1.4106	1.3256	1.2490	1.1768		
	$\mathbf{E}(y^{(r)})$	1.4361	1.2559	1.1264	1.0321	0.9514	0.8793		
I'$_2$	$\mathbf{E}(y)$	5.9470	5.8642	5.8755	5.9297	5.9960	6.0646	90	0.9995
	s.e.	0.1303	0.1179	0.1066	0.0977	0.0902	0.0837		
	sd(y)	1.9102	1.7290	1.5626	1.4321	1.3226	1.2279		
	$\mathbf{E}(y^{(r)})$	1.6902	1.4029	1.2229	1.1226	1.0542	1.0024		
S	$\mathbf{E}(y)$	5.5245	5.5850	5.7328	5.8958	6.0564	6.2099	55	0.9996
	s.e.	0.1344	0.1261	0.1184	0.1119	0.1060	0.1005		
	sd(y)	1.9713	1.8495	1.7367	1.6413	1.5546	1.4743		
	$\mathbf{E}(y^{(r)})$	1.4890	1.2965	1.1735	1.0974	1.0380	0.9865		

properties are not far from those presented in McCulloch (2001), yet for a different period, see Figure 33 in Appendix D. Evans (1998) presents the real yields using U.K. data where the market for real bonds has been active for a longer time. Furthermore, TIPS present issues such as their tax treatment and the inflation indexing lag to determine their payments.[51] The indexing lag and the tax treatment imply that in a strict sense there is no risk-less bond.

As discussed, inflation affects the real yields through consumption growth or the informational content it has for consumption growth. As explained, an inflation shock forecast a decrease in consumption growth, lowering the real yield. The holder of a real bond, as oppose to the holder of a nominal bond, does not have to be compensated for this risk. The real bond will still do well during inflationary times. This explains why the real yield curve tends to slope on average downwards.

13 Time series analysis

Figure 12 depicts the 1 quarter yield implied by specification **H**, $s_{1,t}$ is fixed and $s_{2,t}$ affects Ω. It also has the one from the data. The model is fairly successful capturing the time series dynamics, yet it misses some of changes during the 70s and 80s. For this same configuration, consider the spread and the probability of being in a regime state. Figure 13 depicts the behavior of the spread. The model is moderately successful along this dimension. Figure 14 has the estimate of the probability of the regime state. It is associated with state of/with high volatility. The estimate increases towards the 2000s, it does not seem very plausible given the behavior of the yields at the time.[52]

Moving on to specification **I**, i.e. $s_{1,t}$ and $s_{2,t}$ are independent, Figure 15 presents the short term rate and Figure 17 the spread. It behaves similarly to specification **H**. Yet the short term does a better job capturing the changes during the 70s and 80s. Its behavior depends on the estimates of $\mu_\pi(s_{1,t})$. Figure 19 presents the time series of the regime state estimates, for both regimes, $s_{1,t}$ and $s_{2,t}$. Regime $s_{1,t}$, affecting μ_π, is mainly associated to the changes in

[51] The day the inflation index is set for the payment and the day the actual payment is done are not the same. In the U.S. the lag is 2 months and in the U.K. the lag is 6 months. This feature that is not part of the model. One of the reasons for the existence of the lag is that the inflation index may be revised after it has been published. Usually the real bonds in the U.S. are indexed to the CPI, Consumer Price Index, while those in the U.K. are indexed to the RPI, Retailed Price Index.

[52] This might be the case since the conditional variance of inflation and conditional covariance change are affected with the same regime.

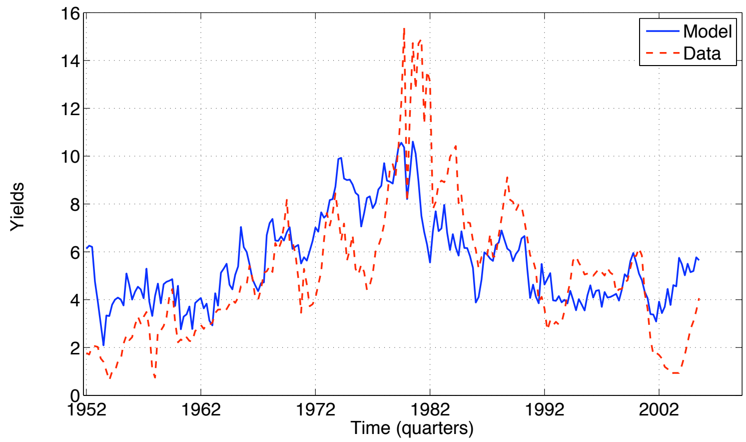

Figure 12: **The short term rate for specification H.**

The quarter yields from the data and those implied by specification **H**, having only on regime affecting Ω.

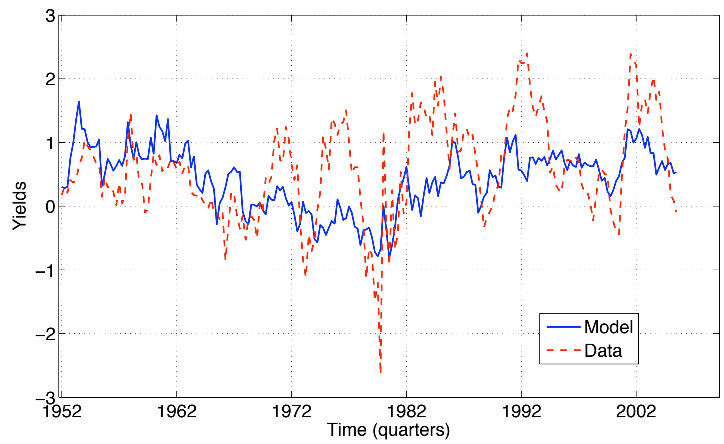

Figure 13: **The yields spread for specification H.**

The spread is 5 year yield minus the 1 year yields. Specification **H**, regime $s_{1,t}$ is fixed and $s_{2,t}$ affects Ω.

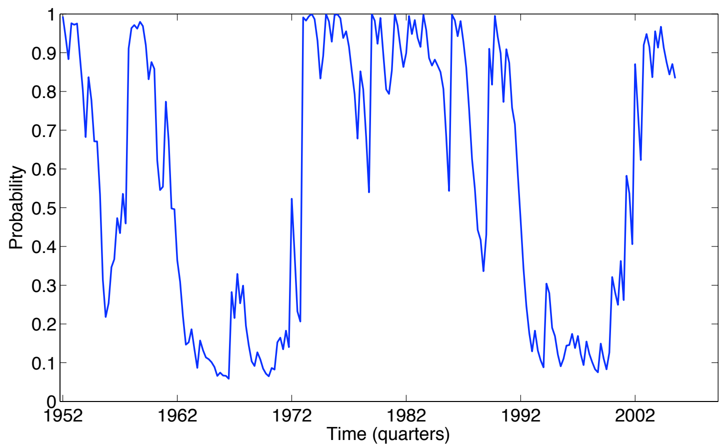

Figure 14: **Estimates of the probability of the regime state, specification H.**

This regime state, $s_{2,t} = 2$, is the one associated to high variance to shocks to inflation and high covariance between the shocks.

monetary policy at the end of the 70s and the 80s, during drastic changes in inflation and the so called monetary experiment. The changes in regime $s_{2,t}$ are very similar to those associated to specification **H**, where $s_{2,t}$ affects Ω.[53]

Consider specification **S**, where $s_{1,t} = s_{2,t}$, Figures 21 and 22 present the short term rate and the spread, respectively. Although the behavior of the short term rate is similar to the rest of the specifications, the spread captures much less of the variation. The estimate of the regime state associated to high mean and high variance in inflation behaves like the top plot in Figure 19. This means that by assuming perfect relationships in the regimes, i.e. $s_{1,t} = s_{2,t}$, the regime $s_{1,t}$ dominates the dynamics. This is why much of the dynamics in $s_{2,t}$ are lost.

The spread, in general for all different specifications, has a fair behavior. The close correlation between the model yields leads to this, since the short term rate captures more variability than the longer maturities. In the model, a trade-off exists between being able to capture the variability of the longer term yields and the variability of the spread. This is because the implied yields are much more correlated among themselves than their data counterparts.

[53]Since fixing $s_{1,t}$ makes it independent to $s_{2,t}$

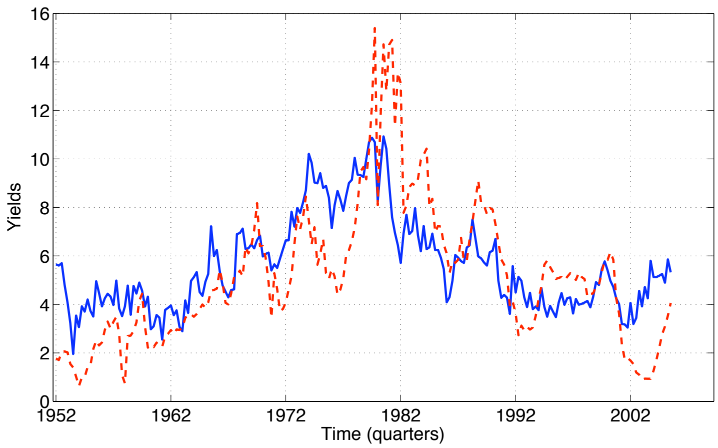

Figure 15: **The short term rate for specification I'.**

The quarter yields from the data and those implied by specification **I**, $s_{1,t}$ independent to $s_{2,t}$.

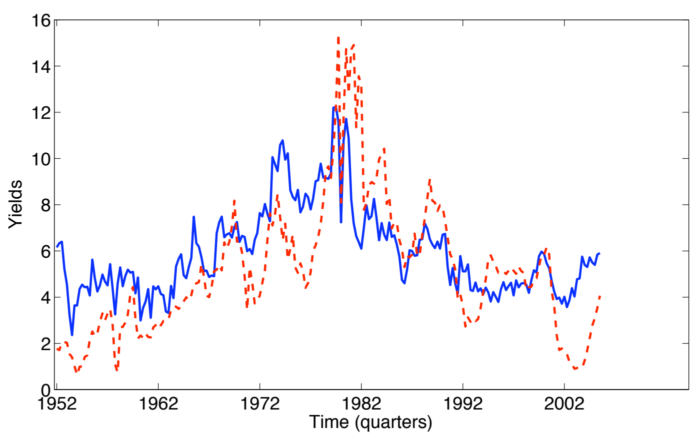

Figure 16: **The short term rate for specification I'$_2$.**

The quarter yields from the data and those implied by specification **I'**$_2$, $s_{1,t}$ independent to $s_{2,t}$.

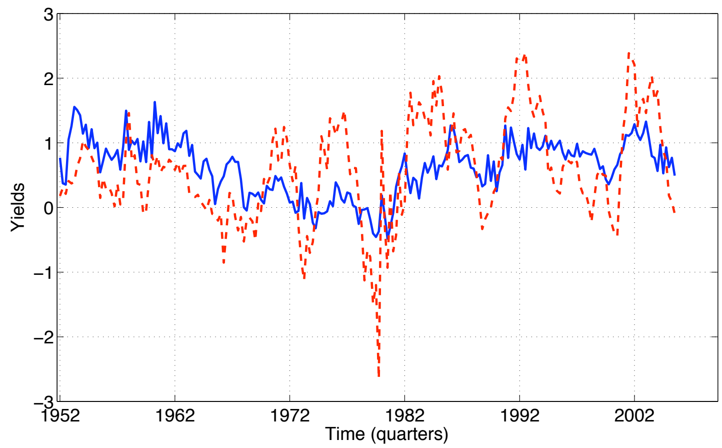

Figure 17: **The yields spread for specification I.**

The spread is 5 year yield minus the 1 year yields. Specification **I**, regime $s_{1,t}$ is independent to $s_{2,t}$.

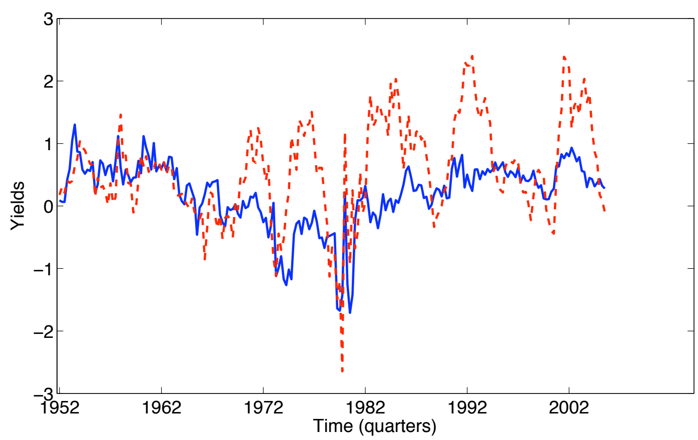

Figure 18: **The yields spread for specification I'**$_2$.

The spread is 5 year yield minus the 1 year yields. Specification **I'**$_2$, regime $s_{1,t}$ is independent to $s_{2,t}$.

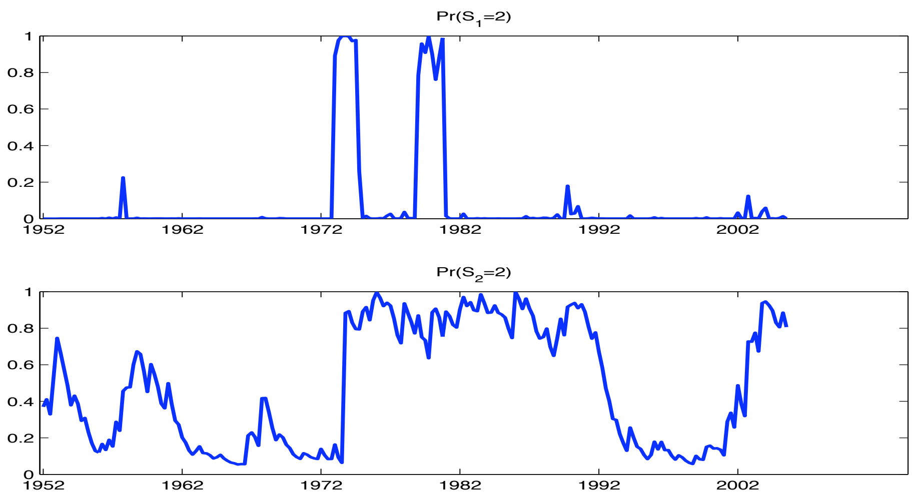

Figure 19: **Estimates of regime states for specification I'.**

The estimates of regime states $s_{1,t}$, associated to μ_π, being in state 2 and regime $s_{2,t}$, associated to Ω, being in state 2 are shown in the top and bottom, respectively. They are assumed to be independent. Changes in the regime states coincide with the inflation episodes of the 70s and the changes in policy in the 80s.

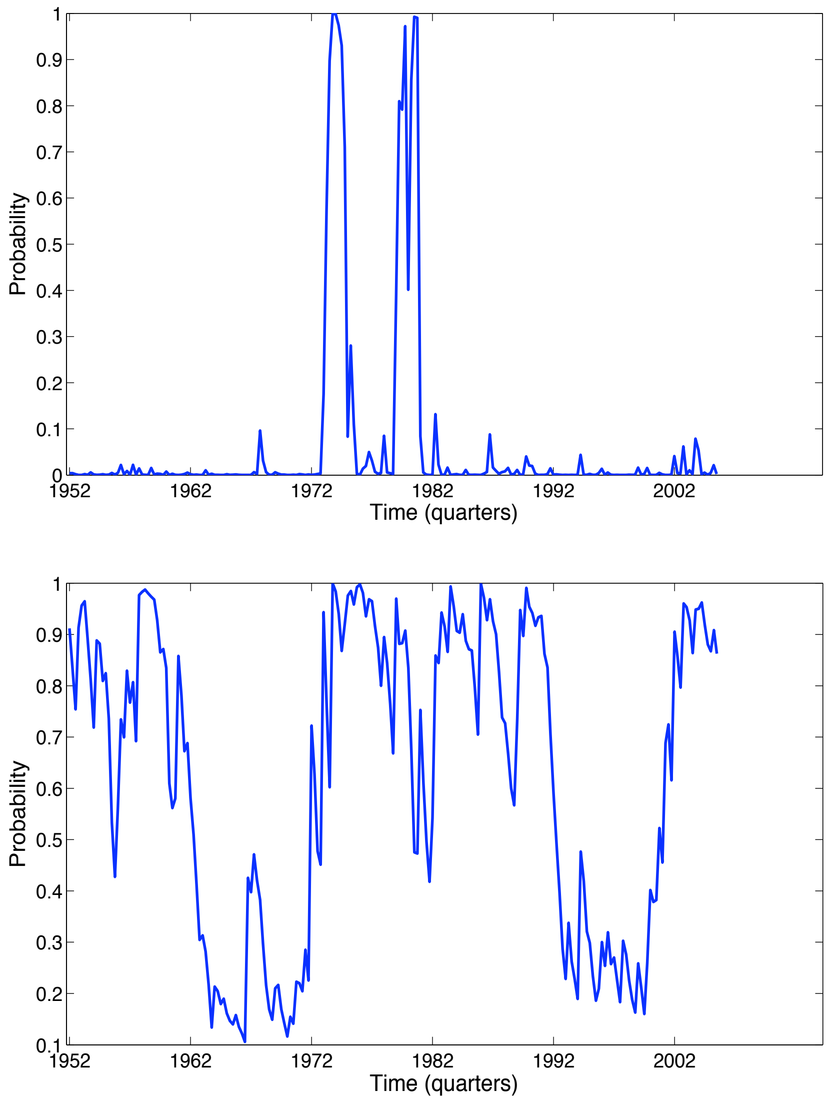

Figure 20: **Estimates of the probability regime states for specification I'$_2$.**

The estimates of the probability regime states $s_{1,t}$, associated to μ_π, being in state 2 and regime $s_{2,t}$, associated to Ω, being in state 2 are shown in the top and bottom, respectively. They are assumed to be independent. Changes in the probability regime states coincide with the inflation episodes of the 70s and the changes in policy in the 80s.

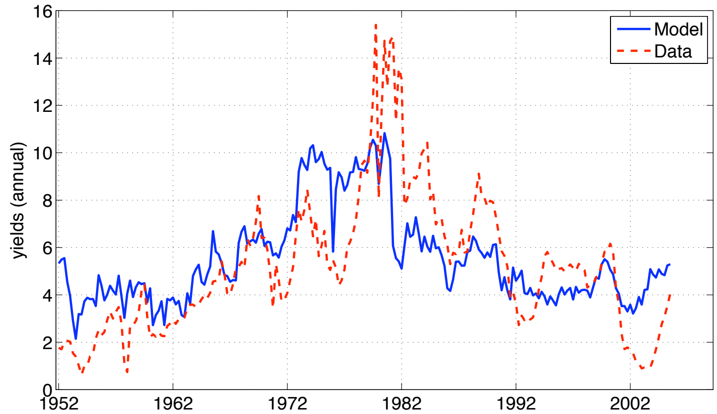

Figure 21: **The short term rate for specification S.**

This is the time series for the short term rate for the case of specification **S**, i.e. $s_{1,t} = s_{2,t}$

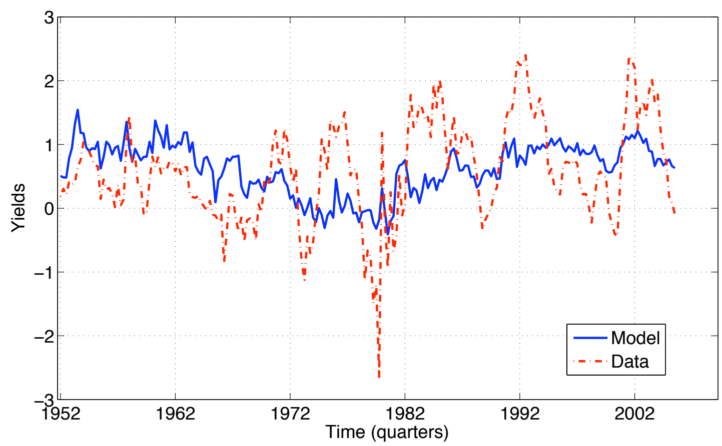

Figure 22: **The yields spread for specification S.**

5 years - 1 year for the case of specification **S**, i.e. $s_{1,t} = s_{2,t}$

14 Principal Components Decomposition

The use of Principal Components Analysis (PCA)[54] started in the term structure of interest rates literature with Litterman and Scheinkman (1991). Their characterization is based on interpreting the first three principal components as movements in the "level," "slope," and "curvature" of the yield curve. These components account for almost all the variability in yields movements, and the proportion of the variance explained decreases with each component. This reduction in the variance explained is captured by the model, although the proportions drop much more rapidly. It was then documented that the aggregate demand shocks are correlated with the first principal component, the level. Changes in monetary policy affect the second principal component, having an effect on the slope, e.g. see Singleton (2006). This is consistent with, for example, Rigobon and Sack (2002), who by estimating a reduced form model conclude that an increase in short-term interest rates results in an upward shift in the yield curve that is smaller at longer maturities. The interpretation of movements in curvature is less clear.

I reexpress the components of yields implied by the model. The idea is to see it as another decomposition with economically interpretable components. The reason why I reexpress the components is because the PCA removes the means from the variables that are analyzed. Thus, recalling $y_t^{(n)} = (A(n) + \mathbf{B}'(n)\mathbf{x}_t + F(n, s_{2,t}) + G(n, s_{1,t}))/n$, we can rewrite $y_t^{(n)} - \mathbf{E}y_t^{(n)}$ as

$$(B_1(n)x_{1,t} + B_2(n)x_{2,t} + (F(n, s_{2,t}) - \mathbf{E}F(n, s_{2,t})) + (G(n, s_{1,t}) - \mathbf{E}G(n, s_{1,t})))/n.$$

Stacking the yields with maturities 1 quarter, 1 year, 2, 3, 4, and 5 years in a vector $\mathbb{Y}_t$, and the coefficients in $\mathbb{B}_1, \mathbb{B}_2, \mathbb{F}(s_{2,t})$, and $\mathbb{G}(s_{1,t})$ to get,

$$\mathbb{Y}_t - \mathbf{E}\mathbb{Y} = \mathbb{B}_1 x_{1,t} + \mathbb{B}_2 x_{2,t} + (\mathbb{F}(s_{2,t}) - \mathbf{E}\mathbb{F}) + (\mathbb{G}(s_{1,t}) - \mathbf{E}\mathbb{G}) \tag{9}$$

Recall that by the PCA provides a decomposition we can express the yield curve at time t as

$$\mathbb{Y}_t - \mathbf{E}\mathbb{Y} = \sum_{i=1}^{6} \alpha_{i,t}\mathbf{v}_i,$$

where $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, and where $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i. In order to compare,

[54] Originally developed by Karl Pearson, among other very well known contributions.

Figure 36 depicts the PCA decomposition for the yields data. Figures 23 and 24 present those implied by specification **H**, i.e. $s_{2,t}$ affect Ω, and specification **S**, for which $s_{1,t} = s_{2,t}$, respectively. The decomposition is somewhat consistent for the former, and closely consistent for the latter.[55]

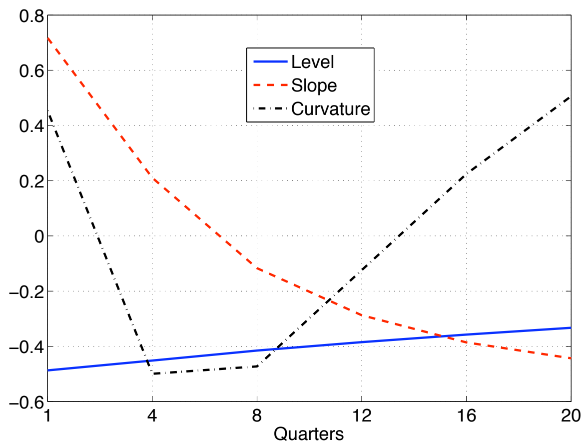

Figure 23: **Principal Components Decomposition for specification H.**

The plots of the first three principal components for the data and for the main model are shown. The interpretations "level," "slope," and "curvature" can be seen. Specification **H** is presented, i.e. with regime $s_{1,t}$ affecting just Ω. It gets right only two of the components.

To see why this is plausible, consider the $\mathbf{B}(\mathbf{n})'\mathbf{x}_t$ function of the yields, Figure 9, and the typical movements of the state variables.

i) Changes in $x_{1,t}$ and $x_{2,t}$ are on average negatively correlated. Naturally, they can sometimes be both positive or both negative. So we would normally see the lines of the plots in Figure 9 on opposite side of the x-axis.

ii) Changes in $x_{2,t}$ account for much of the variations in the yields compared to those $x_{1,t}$. This conforms with the documented result that much of the movements of the yield curve are due to changes in inflation and associated measures.

iii) The component associated to the variance-covariance regime $s_{2,t}$, affects the shorter yield less than the longer ones.

[55] Getting the magnitudes right is aided by the fact that the vectors in the PCA are standardized as described.

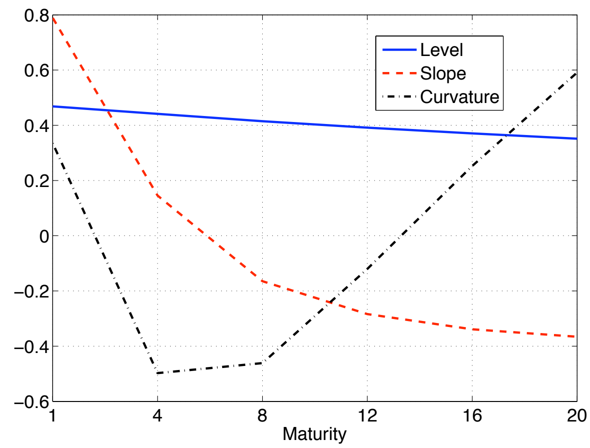

Figure 24: **Principal Components Decomposition for specification S.**

The plot presents the the first three principal components for the model's yield. The interpretations "level," "slope," and "curvature" can be seen. Specification **S** is presented, i.e. with regime $s_{1,t} = s_{2,t}$ affecting both μ_π and Ω. It gets right all of the components.

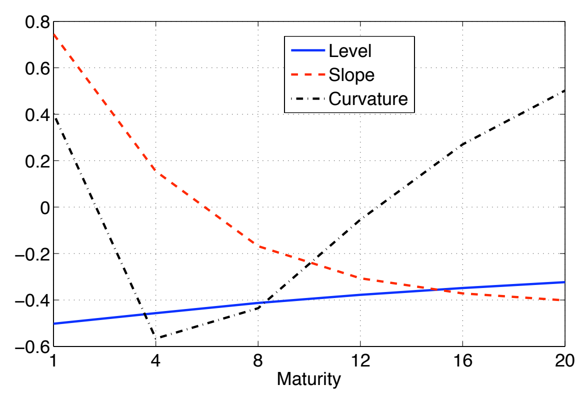

Figure 25: **Principal Components Decomposition for specification I'$_2$.**

The plot presents the the first three principal components for the model's yield. The interpretations "level," "slope," and "curvature" can be seen. Specification **I'**$_2$ is presented, i.e. with regime $s_{1,t} = s_{2,t}$ affecting both μ_π and Ω. It gets right only two of the components.

First, an important part of the changes in yields come from the joint movements in $\mathbf{x}_t$, for the most part these affect the level of the yields. On average, the negative correlation means the $\mathbf{B}_1(n)x_{1,t}$ component will be pulling the yield curve on the opposite direction to $\mathbf{B}_2(n)x_{2,t}$. With typical variations in these components we get changes in the level of the yield curve (see Figure 9), consistent with the behavior of the first principal component. Changes in the regime associated to the mean inflation $(s_{1,t})$ also affect the level of the yield. These are, however, less common.

Second, more drastic changes in the inflation component $x_{2,t}$ can significantly affect the slope, in agreement with the interpretation given to the second principal component. Thus, changes in nominal yields dominated by changes in inflation affect the slope of the yield curve.

Third, variations in the curvature should appear less frequently than those in the level or slope. The curvature varies more drastically when the elements in $\mathbf{x}_t$ go from *both* being negative to *both* being positive or vice versa; which happens with low probability. These type of changes affect the curvature of the yield curve. This is compatible with the low proportion of variance explained by the third principal component, and the type of movements.

To sum up, one can see the model's expression for yields as another decomposition of the movements of the yields which is consistent with the PCA decomposition, and has interpretable economic components. It is remarkable that although the use of information from the yields to obtain the preference estimates in the model was limited to the cross-sectional means, specifications of the model imply a consistent PCA decomposition.

15 Predictability

A central feature of the observed yields is the deviation from the Expectation Hypothesis (EH) in the form of predictability in yields. For example, see Fama and Bliss (1987), Campbell and Shiller (1991), and Cochrane and Piazzesi (2002). It has been documented that expected bonds returns are time-varying. Fama (1984) presents some of the evidence and posits it as a stylized fact models would have to explain. As it is known, these are the same phenomena (see, e.g. Singleton (2006)).

Some models of the term structure of interest rate that are based on the existence of a Stochastic Discount Factor (SDF), e.g. Banzal and Zhou (2001), have been able to capture much of the predictability dynamics in the yields. However, consumption-based term structure models have been less explored along this dimension.[56] I will show that the model has important implications for predictability.

Table 15 conveys two central facts. First, the model captures the magnitude of the (log) holding period returns (hpr) (some measures are slightly off from their standard errors). Recall that the (log) holding period return, hpr, is given by $hpr_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)}$. Second, for the (log) implied by the model, as in the data, on average there is no difference between holding bonds of different maturities for a period, a year for this case. In this sense, the EH holds on average, for the model as well as for the data.

Next, I consider the predictability regression tests as implemented in Cochrane (2001), to measure the success of the model along this dimension. I then proceed with the regression tests as in Cochrane and Piazzesi (2002).

Table 15 shows the results for the initial predictability tests. For the change in yields regressions the EH predicts a coefficient of one for the coefficient b. In the data this tends to hold the bigger N is. Specification **H** gets the tendency of the coefficient in the opposite direction. While, specification **I** is more successful, it starts conforming with the EH as N increases. In specification **S**, the EH holds for all values of N. The model in the short run presents some predictability whereas the data does not. This relates to the high variability of the short term rate not captured by the model. They all tend to increase their predictability as N increases, as reflected by the adjusted R^2.

For the holding period returns regressions, see Table I, the EH predicts a value of zero for the coefficient b. Specification **H**, maintains a lower coefficients and adjusted R^2's. For specification **I**, the estimates are closer to the data. This is consistent with the results for the change in yields regressions. Specification **S** gets the magnitudes in the wrong direction. It is not completely clear why the estimates for the holding period returns regressions are far relative to the data, in contrast to the change in yields regressions.

In sum, by the properties of consumption growth, inflation, and the estimates, predictability would be expected to hold only to an extent. In other words, there is a limited content of predictability in inflation and consumption growth, relative to the predictability in yields.[57] Yet, while specification **H**

[56] One exception is Wachter (2006). She extends the Campbell and Cochrane's (1999) habits model to obtain a term structure of interest rate.

[57] It will also depend on the functional forms that connect the data.

Table 5: **Mean holding period returns.**
These are mean holding period return, the corresponding standard errors, and the standard deviations of logarithm Holding Period Return (hpr) for maturities 1 to 4 years. The holding period return is one year. The data are from 1952:2 to 2005:4. Data source: CRSP. The specification **H**, $s_{1,t}$ is fixed and $s_{2,t}$ affects Ω. The specification **I** stands for independent regimes $s_{1,t}$ and $s_{2,t}$, affecting Ω and μ_π respectively. The specification **S** stands for $s_{1,t} = s_{2,t}$ affecting both Ω and μ_π.

	Maturity	$\mathbf{E}(hpr_{t+1}^{(n)})$	std.error	std. dev.
Data	1	5.9702	0.2582	3.7505
	2	6.2323	0.3245	4.7135
	3	6.3392	0.3963	5.7672
	4	6.1543	0.4566	6.6323
H	1	5.8889	0.1394	2.0246
	2	6.1007	0.1822	2.6464
	3	6.3161	0.2226	3.2330
	4	6.5008	0.2581	3.7485
I'	1	5.8604	0.1274	1.8500
	2	6.0893	0.1744	2.5331
	3	6.3058	0.2187	3.1762
	4	6.5231	0.2550	3.7047
I'$_2$	1	5.8911	0.1375	1.9966
	2	5.9311	0.1768	2.5676
	3	6.0169	0.2123	3.0838
	4	6.0852	0.2443	3.5480
S	1	5.8869	0.1501	2.1810
	2	6.1178	0.1966	2.8561
	3	6.3535	0.2433	3.5342
	4	6.5458	0.2883	4.1883

Table 6: **Change in yields predictability regressions.**
The period is from 1952:2 to 2005:4. Source: CRSP. These regressions are for the following specifications: **H** is the case for which $s_{1,t}$ is fixed and $s_{2,t}$ affects Ω. **I** is the case when $s_{1,t}$ and $s_{2,t}$ are independent. **S** is the case when $s_{1,t} = s_{2,t}$ are the same. s.e. stands for standard errors and adj for adjusted. Under the expectation hypothesis the value of b should be 1. The N indicates the number of periods after t the future yield is considered, it is in years. $y_{t+N}^{(1)} - y_t^{(1)} = a + b(f_t^{(N \to N+1)} - y_t^{(1)}) + \epsilon_{t+N}$.

N	a	$s.e.(a)$	b	$s.e.(b)$	adj. R^2
Data					
1	0.0015	0.1342	0.0813	0.1669	-0.008
2	-0.2284	0.1939	0.3800	0.1599	0.017
3	-0.5754	0.2187	0.6227	0.1521	0.068
4	-0.6109	0.2268	0.8077	0.1531	0.115
H					
1	-0.2801	0.0949	0.8574	0.2031	0.0742
2	-0.5167	0.1335	0.6939	0.1566	0.0829
3	-0.6237	0.1588	0.7187	0.1345	0.1201
4	-0.6679	0.1686	0.6047	0.1159	0.1170
I'					
1	-0.1271	0.0844	0.5414	0.1866	0.0214
2	-0.2829	0.1299	0.5974	0.1604	0.0401
3	-0.5463	0.1610	0.6339	0.1421	0.0855
4	-0.6292	0.1703	0.5611	0.1205	0.0946
I'$_2$					
1	-0.0226	0.0688	1.0120	0.1829	0.1236
2	-0.1362	0.0902	1.0095	0.1432	0.1911
3	-0.2346	0.1035	0.8913	0.1225	0.2044
4	-0.2836	0.1090	0.7342	0.1069	0.1892
S					
1	-0.2847	0.1023	0.9698	0.2527	0.0614
2	-0.6193	0.1597	1.0273	0.2064	0.1035
3	-0.9415	0.1892	1.0473	0.1702	0.1543
4	-1.0577	0.2026	0.9074	0.1440	0.1634

follows somewhat closely the EH, the predictability in specification **I** ($s_{1,t}, s_{2,t}$ independent) is in the right direction. An increase in the difference between the forward rate and the short rate predicts an increase in future's (excess) yields or holding period return. Assuming independence in regimes affecting the inflation mean and the variance-covariance matrix contributes to getting both closer magnitudes and patterns for the basic predictability regression tests.

Moving on to the Cochrane and Piazzesi (2002) tests, we have that their initial regression is given by:

$$hpr_{t+1}^{(n)} - y_t^{(1)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_2^{(n)} f_t^{(2)} + ... + \beta_5^{(n)} f_t^{(5)} + \epsilon_{t+1}^{(n)},$$

recalling that $f_t^{(k)}$ denotes the forward rate. Their tests are based upon a generalization of the holding period return regressions. The idea is that no additional variables should improve the forecast capabilities provided that the EH holds. This regression for the model data presented collinearity. This is not surprising given that small number of state variables that are part of the model. Simplifying the test to sidestep the collinearity problem the regression

$$hpr_{t+1}^{(n)} - y_t^{(1)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_3^{(n)} f_t^{(3)} + \beta_5^{(n)} f_t^{(5)} + \epsilon_{t+1}^{(n)},$$

was performed instead, dropping two of the intermediate forwards rates. Figure 26 presents the tent-shaped functions for specification **I**, i.e. $s_{1,t}$ independent of $s_{2,t}$, and Figure 28 presents the tent-shaped functions for the specification **S**, i.e. $s_{1,t} = s_{2,t}$

Table 7 presents the estimates for the simplified Cochrane Piazzesi regressions. As in the simple regressions the low values of the adjusted R^2's are as expected in the sense that relatively to the yields' predictability, the information contained in consumption growth and inflation is small, as mentioned. The increase of the standard errors as the maturity increases reflects the difficulty the model faces measuring elements in the long run. It is certainly remarkable that the magnitude of the coefficients are close and have the same pattern.

Recall that only the cross sectional averages of the yields were used to obtain the preference parameters. In other words, the model is not heavily relying on the spline used to obtain the data yields. However, Singleton (2006) argues that the tent-shaped function obtained in Cochrane and Piazzesi (2002) depends on the smoothing construction of the yields data. Thus, the results presented here

Table 7: **Simplified Cochrane-Piazzesi regressions.**
The simplified Cochrane Regression $hpr_{t+1}^{(n)} - y_t^{(1)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_3^{(n)} f_t^{(3)} + \beta_5^{(n)} f_t^{(5)} + \epsilon_{t+1}^{(n)}$. The period is from 1952:2 to 2005:4. Source: NIPA and CRSP.

n	β_0	s.e.	β_1	s.e.	β_3	s.e.	β_5	s.e.	adj. R^2
Data									
1	-1.237	0.292	-0.670	0.112	1.453	0.2210	-0.585	0.170	0.24
2	-1.876	0.527	-1.453	0.202	3.126	0.5000	-1.395	0.306	0.27
3	-2.816	0.716	-2.163	0.274	4.541	0.5430	-1.993	0.516	0.30
4	-3.670	0.899	-2.681	0.345	5.097	0.6820	-1.958	0.523	0.27
H									
1	-5.005	2.615	0.154	0.505	-1.959	1.677	2.457	1.602	0.024
2	-9.327	4.666	0.259	0.722	-3.670	2.992	4.627	2.858	0.028
3	-13.145	6.40	0.368	0.990	-5.271	4.104	6.614	3.920	0.030
4	-16.491	7.880	0.578	1.220	-6.738	5.053	8.402	4.826	0.031
I'									
1	2.225	1.800	-1.252	0.278	3.724	1.149	-2.658	1.101	0.105
2	3.027	3.263	-2.135	0.504	6.135	2.083	-4.204	1.995	0.102
3	3.804	4.549	-2.900	0.702	8.168	2.904	-5.501	2.782	0.099
4	4.514	5.669	-3.555	0.875	9.868	3.619	-6.581	3.467	0.095
I'$_2$									
1	4.2940	2.5980	-0.1360	0.3470	1.8540	1.5750	-2.3140	1.6230	-0.0000
2	7.5160	4.5000	-0.2440	0.6010	3.2040	2.7280	-3.9860	2.8120	-0.0010
3	9.9850	6.0340	-0.2950	0.8060	4.1410	3.6580	-5.1960	3.7710	-0.0010
4	11.9080	7.3080	-0.3120	0.9760	4.8050	4.4300	-6.0950	4.5660	-0.0010
S									
1	1.366	3.729	-0.614	0.391	2.015	2.040	-1.492	2.184	0.003
2	1.913	6.868	-1.204	0.721	3.592	3.757	-2.478	4.023	0.006
3	2.215	9.669	-1.743	1.015	4.939	5.289	-3.263	5.663	0.007
4	2.367	12.172	-2.231	1.278	6.101	6.658	-3.905	7.128	0.007

support the robustness of the results in Cochrane and Piazzesi (2002).[58]

In sum, the role of the regimes and the way they are specified in the model are crucial to obtain some of the patterns that appear in the simplified Cochrane and Piazzesi regressions, which are largely successful in the model. In their paper, Cochrane and Piazzesi (2002), conclude that the tent-shaped function suggests it is one factor driving the predictability in yields. The structure of the model and the estimates of the different specifications imply that the regime $s_{2,t}$ captures an important component of the predictability dynamics in the yields. Thus, our model suggests, that this is the one factor driving predictability. This regime can be interpreted as a time-varying change in risk.

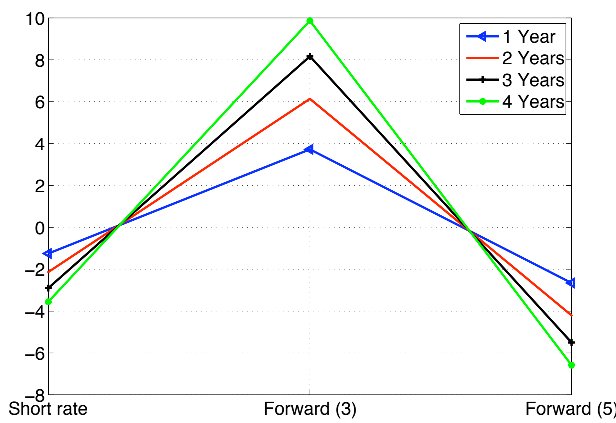

Figure 26: **The Cochrane Piazzesi tent-shaped plots for specification I.**

These are the $\beta_1^{(n)}, \beta_3^{(n)}$ and $\beta_5^{(n)}$ of the regression. The plot shows the coefficients from regression (10) using the yields implied by the model **I'**, i.e. $s_{1,t}$ and $s_{2,t}$ are independent.

[58]The comparison is not strictly direct since some of the forwards were dropped, for reasons already explained.

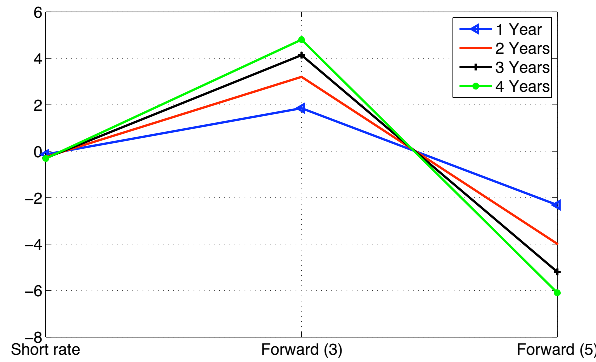

Figure 27: **The Cochrane Piazzesi tent-shaped plots for specification $\mathbf{I}_2$.**

These are the $\beta_1^{(n)}, \beta_3^{(n)}$ and $\beta_5^{(n)}$ of the regression. The plot shows the coefficients from regression (10) using the yields implied by the model $\mathbf{I'}_2$, i.e. $s_{1,t}$ and $s_{2,t}$ are independent.

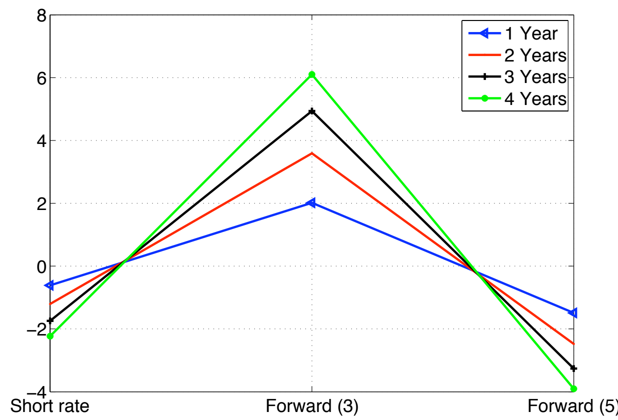

Figure 28: **The Cochrane Piazzesi tent-shaped plots for specification S.**

These are the $\beta_1^{(n)}, \beta_3^{(n)}$ and $\beta_5^{(n)}$ of the regression. The plot shows the coefficients from regression (10) using the yields implied by the model S, i.e. $s_{1,t}$ and $s_{2,t}$ are the same.

16 Conclusions

Hayek (1945) could have probably foretold the difficulties consumption-based asset pricing models have faced in the past decades. Obtaining yields only from aggregate consumption would hardly be a simple feat, since prices already summarize and convey a remarkable amount of information. Besides consumption additional variables need to be considered in order to understand: i) the extent to which changes in macroeconomic variables are perceived as transient or long lasting, ii) the effects shocks have on the variables, in particularly their intertemporal dynamics, iii) the various measures of risk, and, thus, iv) their effects on yields.

This paper proposes a consumption-based asset pricing model to analyze the relationship between consumption growth, inflation, and yields. The inclusion of regime switching and the exact specification play a key role. Generally the idea of introducing regimes is to capture changes in the relationships both in the macroeconomic and financial variables. Yet, in the model, it is the combination of recursive utility and the persistence in regimes that produce relevant implications for the behavior of the yields.

The estimation of the state-space uses consumption growth and inflation as macroeconomic variables. While the estimation of the preference parameters uses only the average yield curve. Yet, it is remarkable the extent to which the model conforms with an important set of the yields' stylized facts. These were not, however, perfectly consistent across the specifications.

The estimation and measurement of parameters entail econometric challenges. First, for example, an unrestricted transition matrix[59] implies a state-space that quickly becomes an intractable object to estimate. The difficulty of this estimation is reminiscent of the problems measuring the cross responses of other variables. Thus, in the case of regimes switching it seems a convenient solution to estimate the polar cases, as I do.[60] Second, the matrix ϕ_x plays a triple role: i) it measures the short term and the long run responses of the variables; ii) it ties the cross-sectional behavior with the time series dynamics of yields; and iii) it defines the short and long run prices of risk. Third, the model hinges on measuring the temporal distribution of risk. Increasing the parameter that measures this risk allows it to play an important role. On the one hand, it is hard to measure. On the other, accounting for it delivers a model with relevant

[59] Specifically, an arbitrary relationship between $s_{1,t}$ and $s_{2,t}$

[60] The number of regime states considered is relevant to the independence assumption.

empirical implications. The crux is then its validity and the macroeconomic variables behind it. At least, two further issues arise.

First, I use the same parameter to measure risk aversion and the intertemporal distribution of risk. They should not be necessarily the same. This is analogous to the familiar case of power utility, in which there is only one parameter measuring risk aversion and the elasticity of intertemporal distribution. These two concepts that are in principle unrelated. Second, what the econometrician might end up measuring might play a more significant role. For example, an agent observes a negative macroeconomic shock. Assume he then reacts to it and hedges this shock to the extent possible. His consumption decreases less than what he might consumed had there not been a hedging mechanism. The economist then collects the realized data. Although, the macroeconomic indicator was at the time informative, the correlation might decrease ex-post by the nature of the time-varying risk-sharing mechanism.[61] The difference between the expected (ex-ante) versus realized (ex-post) measurement of the intertemporal distribution of risk might be substantial. Concerns like this should motivate different ways of measuring risks.

I see the following relevant future exercises. First, the possibility to estimate more general specifications by adding further economic information relevant to the agent. Additional economic information, e.g. corporate earnings, inflation surveys, futures on the short term rate or a sensible function of these variables might potentially contain information regarding future consumption growth prospects. A parallel extension is to use additional variables minimizing the number of additional parameters to be estimated. This might involve varying the exact specification of the state-space.

Second, the possibility to entertain values different from one for ψ. Assuming that the elasticity of intertemporal substitution ψ is not one. not only simplifies the formulas to obtain the bond prices but it sidesteps the need to estimate wealth, an unobservable variable. Thus, there would be a need to estimate the wealth. This assumption is not innocuous and might be hiding significant effects. This could potentially give the model further information with relevant implications on the value of the estimates, the dynamics of the variables, and, thus, the behavior of the yields.

Third, the possibility of positing a utility function that uses different pa-

[61] Recall that it is only macroeconomic risk that the agent is compensated for. Thus, naturally, measuring intertemporal risk should account for this. Also, the change in the risk-sharing mechanism through time can be motivated by time-varying investment sets.

rameters to measure risk aversion and the intertemporal distribution of risk. Independent evidence on the magnitude of the parameter measuring the latter risk would be needed. This idea would not be exempt from the inherent difficulties measuring different risks and their properties. Some of these and related issues are, to an extent, discussed in the appendices.

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A Value Function

Here I solve for the value function drawing results from Hansen (2007), Hansen (2006), and Hansen, Heaton and Li (2008). To simplify the notation I give the functions f their linear notations and assume independence among the regimes. The general case is similar.

$$\begin{aligned} \mathbf{z}_{t+1} &= (\mu_{\Delta c} \;\; \mu_\pi(s_{1,t}))' + \mathbf{x}_t + \epsilon_{t+1} \\ \mathbf{x}_{t+1} &= \phi_x \mathbf{x}_t + K\epsilon_{t+1}, \end{aligned} \tag{10}$$

The shock ϵ_{t+1} affecting both equations is normally distributed with mean $(0 \;\; 0)'$, and variance-covariance matrix $\Omega(s_{2,t})$. Recall, $s_{1,t}$ is independent to $s_{2,t}$. The equation the continuation value needs to satisfy.

$$v_t \quad = \quad (1-\beta)c_t + \beta(1-\gamma)^{-1}\log \mathbf{E}_t(\exp((1-\gamma)v_{t+1})).$$

Conjecture that the value function is of the form:

$$v_t = a + \mathbf{b}'\mathbf{x}_t + c_t + f(s_{2,t})$$

Consider the expression $\mathbf{E}_t \exp((1-\gamma)v_{t+1})$,

$$\begin{aligned} &\text{Constant at time } t \qquad \exp((1-\gamma)(a + \mathbf{b}'\phi_x\mathbf{x}_t + c_t + \mu_{\Delta c} + e_1'\mathbf{x}_t)) \\ &\text{Stochastic at time } t \qquad \exp((1-\gamma)(\eta'\epsilon_{t+1} + f_1 + f_2 s_{2,t+1})) \end{aligned}$$

where $\eta' = \mathbf{b}'K + e_1'$. Thus, due to independence, to solve for the value function we have the following equations:

$$
\begin{aligned}
a &= \beta\mu_{\Delta c} + \beta a \\
a &= \frac{\beta}{(1-\beta)}\mu_{\Delta c} \text{ with regimes in Omega} \\
a &= \frac{\beta}{(1-\beta)}\mu_{\Delta c} + \frac{\beta}{(1-\beta)}(1-\gamma)\eta'\Omega\eta/2 \text{ with no regimes in Omega} \\
c_t &= (1-\beta)c_t + \beta c_t \\
\mathbf{b}' &= \beta(\mathbf{b}'\phi_x + e_1') \\
\mathbf{b}' &= \beta e_1'(I - \beta\phi_x)^{-1} \\
\mathbf{b} &= \beta(I - \beta\phi_x')^{-1}e_1 \\
f_1 + f_2 s_{2,t} &= \frac{\beta}{(1-\gamma)}\log \mathbf{E}_t \exp\left((1-\gamma)(f_1 + f_2 s_{2,t+1})\right) + \frac{\beta}{2}(1-\gamma)\eta'\Omega(s_{2,t})\eta
\end{aligned}
$$

The last equation can be seen as a discrete Ricatti equation, it is solved numerically for f_1 and f_2, more generally for f. This set of equations define the continuation value, and by construction it satisfies it.

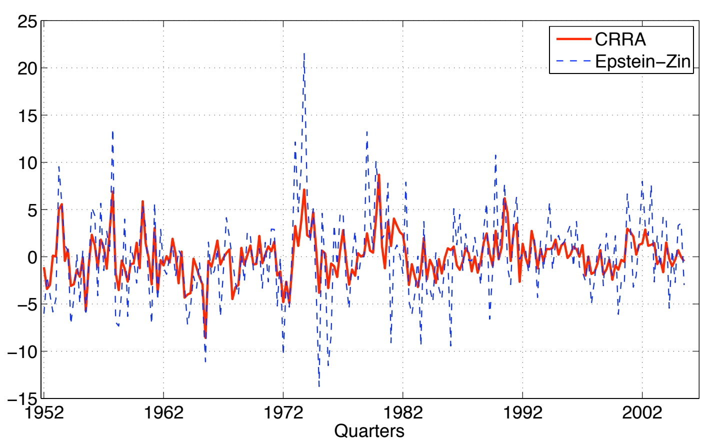

Figure 29: **Realized time series of the (log) real S.D.F.**

The plot presents the realized time series for the (log) real Stochastic Discount Factor under Epstein-Zin and under CRRA utility, $\gamma = 5$ and $\beta = 0.995$. The x-axis is the time, in quarters.

B SDF, a second interpretation

Now consider an alternative expression for the (log) nominal SDF. To this end we have the following definitions

$$
\begin{aligned}
\kappa &\equiv \log\beta - \gamma\mu_{\Delta c} \\
\xi &\equiv -e_1' - e_2' \\
\sigma_1(s_{2,t}) &\equiv \left(-\gamma e_1' - e_1'(\gamma-1)\beta(I-\phi_x)^{-1}K - e_2'\right)\Sigma(s_{2,t}) \\
\sigma_2(s_{2,t+1}, s_{2,t}) &\equiv \qquad -(\gamma-1)\left(f(s_{2,t+1}) - \beta^{-1}f(s_{2,t})\right)
\end{aligned}
$$

where $\Omega(s_{2,t}) = \Sigma(s_{2,t})\Sigma(s_{2,t})'$, $e_1' = (1 \quad 0)$ and $e_2' = (0 \quad 1)$. I can then rewrite the (log) nominal SDF as $\kappa + \xi'\mathbf{x}_t + \sigma_1(s_{2,t})\zeta_{t+1} + \sigma_2(s_{2,t+1}, s_{2,t}) - \mu_\pi(s_{1,t})$, where $\zeta_{t+1} \sim \mathcal{N}(0, I)$.

The constant factor κ depends on the risk aversion coefficient γ, the subjective discount factor β, and the mean consumption growth $\mu_{\Delta c}$. The factor loadings ξ have a short term effect, through -1 for $x_{1,t}$ and -1 for $x_{2,t}$.

The market price of risk is given by $\sigma_1(s_{2,t})$. It has three components, one depends on $-\gamma\Sigma(s_{2,t})$ capturing the short run effect. The second is $(\gamma-1)(I-\beta\phi_x)^{-1}\Sigma(s_{2,t})$, accounting for the cross and long run effects. Its importance grows with the persistence of $\mathbf{x}_t$. Last, $-e_2'\Sigma(s_{2,t})$, takes into account the shock to inflation.

Changes in $s_{2,t}$ have a long run effect denoted $\sigma_2(s_{2,t+1}, s_{2,t})$. More persistence, as measured by the transition probabilities $r_{i,i}$, will make the difference $f(s_{2,t+1}) - \beta^{-1}f(s_{2,t})$ larger.

Recall the notation: $f(s_{2,t+1}) = f_1 + f_2 s_{2,t+1}$ in the two regime states case.

The price of a "jump" from $s_{2,t} = 0$ to $s_{2,t+1} = 1$ (from $s_{2,t} = 1$ to $s_{2,t+1} = 0$) is given by $(1-\gamma)f_2$ (respectively, $(-(1-\gamma)\beta^{-1}f_2)$). The more persistent the regime $s_{2,t}$ is, the bigger in absolute value the jump in the price is. If there is no persistence, i.e. $r_{i,i} = 1/2$ for $i = 1, 2$, the price of a jump lowers. These results are a direct consequence of the properties of the continuation value. Finally, the regime $s_{1,t}$ in $\mu_\pi(s_{1,t})$ affects the (log) SDF linearly.

C Bond Pricing details

This section presents how to solve for the bond prices. I will follow again the case of the specification in the last section. The proofs for the general case are analogous. Conjecture a price function of the form.

$$P_t^{(n)}(\mathbf{x}_t, s_t) = \exp(-A(n) - \mathbf{B}(n)'\mathbf{x}_t - F(n, s_{t,2}) - G(n, s_{t,1}))$$

Thus, recall that

$$\begin{aligned} v_{t+1} &= a + \mathbf{b}'x_{t+1} + c_{t+1} + f_1 + f_2 s_{2,t+1} \\ m_{t+1} &= \log\beta - \gamma\Delta c_{t+1} + (1-\gamma)(\hat{v}_{t+1} - \beta^{-1}\hat{v}_t) - \pi_{t+1} \end{aligned}$$

where $\hat{v}_{t+1} = v_{t+1} - c_{t+1}$. It follows that

$$\begin{aligned} (\hat{v}_{t+1} - \beta^{-1}\hat{v}_t) &= a + \mathbf{b}'x_{t+1} + f_1 + f_2 s_{2,t+1} \\ &\quad -\beta^{-1}a - \beta^{-1}\mathbf{b}'x_t - \beta^{-1}f_1 - \beta^{-1}f_2 s_{2,t} \\ &= \mathbf{b}'(\phi_x - \beta^{-1}I)x_t + \mathbf{b}'\phi_x K\epsilon_{t+1} + a(1-\beta^{-1}) + \\ &\quad + f_1(1-\beta^{-1}) + f_2 s_{2,t+1} - \beta_2^{-1} f_2 s_{2,t} \end{aligned}$$

Now since

$$\mathbf{E}\left(\exp(m_{t+1}^{(r)} - \pi_{t+1})P_{t+1}^{(n-1)}(\mathbf{x}_{t+1}, s_{t+1})/P_t^{(n)}(\mathbf{x}_t, s_t)\right) = 1,$$

by the conjecture of the price, the S.D.F., and regimes independence; we obtain the following recursive relationships:

$$\begin{aligned} A(n) &= A(n-1) - \log(\beta) - \gamma e_1'\mu_1 + (1-\gamma)a(1-\beta^{-1}) \\ &= A(n-1) - \log(\beta) + \mu_{\Delta c} \text{ with regime in Omega} \\ &= A(n-1) - \log(\beta) + \mu_{\Delta c} + \frac{(1-\gamma)^2}{2}\eta'\Omega\eta \text{ w/ no regimes in Omega} \\ \mathbf{B}(n) &= \phi_x'\mathbf{B}(n-1) + \gamma e_1' + (1-\gamma)\mathbf{b}'(\phi_x - \beta^{-1}I) - e_2' \\ &= \phi_x'\mathbf{B}(n-1) + e_1 + e_2 \text{ with and without regimes} \\ F(n, s_{2,t}) &= -\log\left(\mathbf{E}_t(\exp(-F(n-1, s_{2,t+1}) + 0.5\psi(n-1)'\Omega(s_{2,t})\psi(n-1)))\right) \end{aligned}$$

where $\psi(n-1) \equiv -\gamma e_1' + (1-\gamma)\mathbf{b}'K - \mathbf{B}(n-1)'K - e_2 = -\gamma e_1' + (1-\gamma)\beta e_1'(I - \beta\phi_x)^{-1}K - \mathbf{B}(n-1)'K - e_2$. Recall that $A(0) = 0$, $\mathbf{B}(0) = \mathbf{0}$ and $F(0, s_{2,t}) = 0$ for all s_t since $P_t^{(0)}(\mathbf{x}_t, s_t) = 1$ for all $\mathbf{x}_t$ and s_t, which defines the initial conditions. These can be interpreted as discrete Ricatti equations.

Note that the construction of the price for the real bond is totally analogous, except that, naturally, the inflation is excluded (equivalently set $e_2' = (0\ \ 0)$).

For a general relationship in the regimes, the construction is similar except that the regime is expanded $\mathbf{s}_t \equiv (s_{1,t}, s_{2,t})$ and thus the form of the bond price is of the form:

$$P_t^{(n)}(\mathbf{x}_t, s_t) = \exp(-A(n) - \mathbf{B}(n)'\mathbf{x}_t - H(n, \mathbf{s}_t)).$$

D Data details

This section is not meant not be self-contained but rather as a complement of the text. The plots for time series of consumption growth, inflation, yields and real yields are presented. By visual inspection, the plot for inflation justifies the exploration of the regime in the mean inflation.

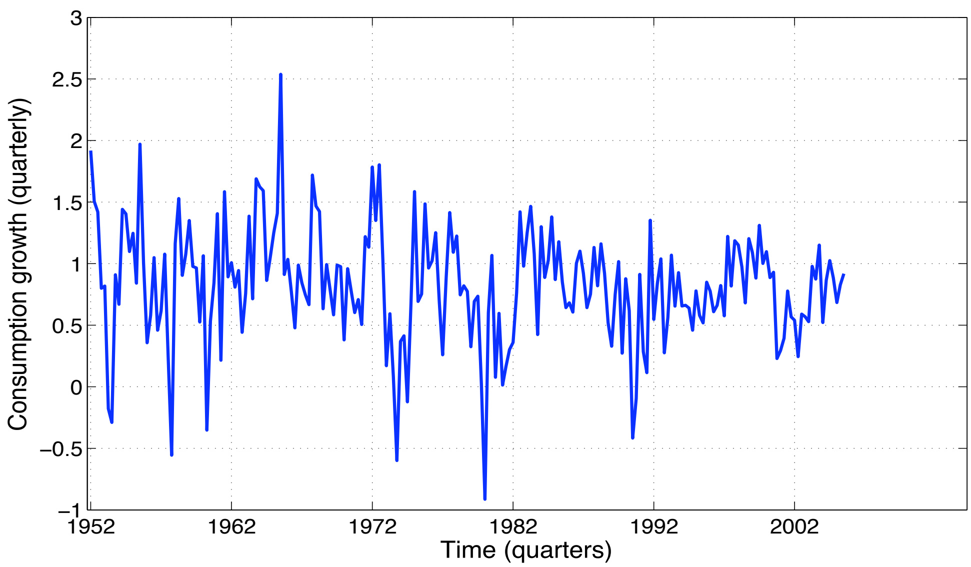

Figure 30: **Time series for real consumption growth.**

The consumption considered is for nondurables and services. The population growth is assumed to be constant an zero. Thus, the consumption growth is not adjusted. Source: NIPA.

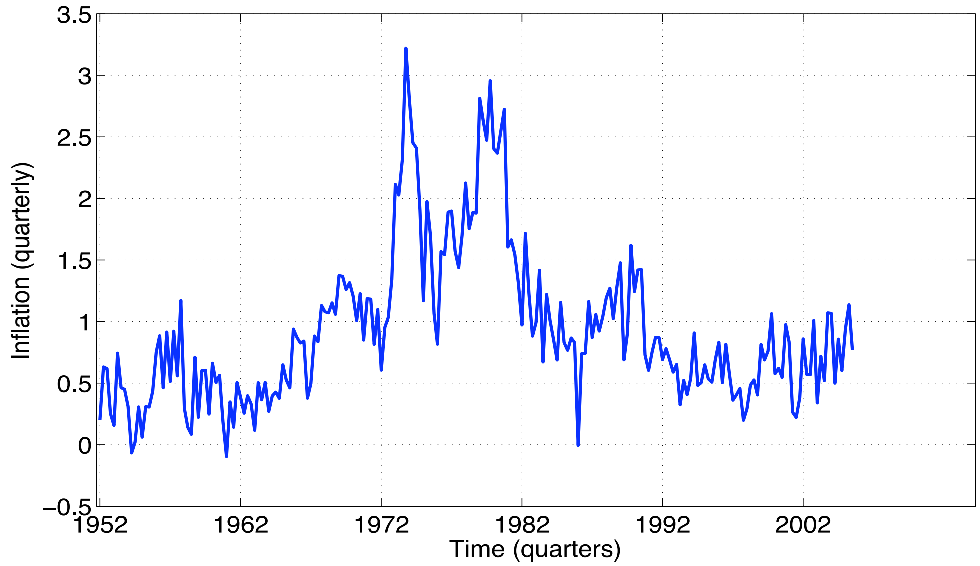

Figure 31: **Time series for inflation.**

E Estimation details

This section presents estimates for the following specifications of the model:

1.N̲o regimes: There are no regimes affecting the state-space.

2.H̲eteroscedasticity: regime $s_{1,t}$ is fixed, i.e. μ_π is constant, and regime $s_{2,t}$ affects $\Omega(s_{2,t})$

3.I̲ndependent: regime $s_{1,t}$ affects $\mu_\pi(s_{1,t})$, and $s_{2,t}$ affects $\Omega(s_{2,t})$ and regime $s_{1,t}$ and $s_{2,t}$ are independent.

4.S̲ame: $s_{1,t}$ affects $\mu_\pi(s_{1,t})$ and $\Omega(s_{1,t})$. In other words, $s_{1,t} = s_{2,t}$.

5.Only one regime associated to ϕ_x.

6.A model with a third variable, labor income growth.

It also presents detailed descriptions of the estimation methods used in the paper.

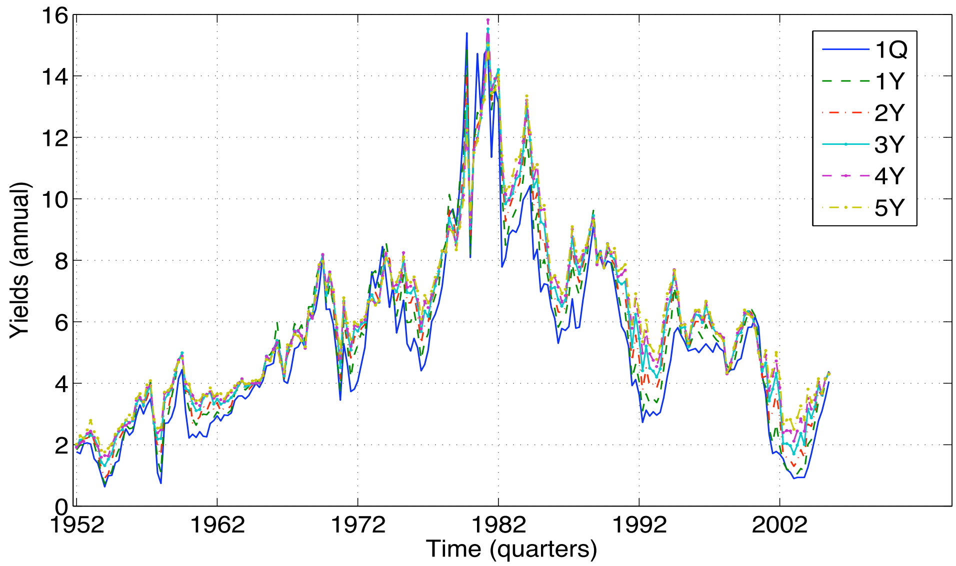

Figure 32: **Time series for the yields, all 6 maturities.**

These are the time series for the yields of all the 6 maturities used in the paper: 1 Q, 1 year, 2, 3, 4, and 5 years. Source: CRSP.

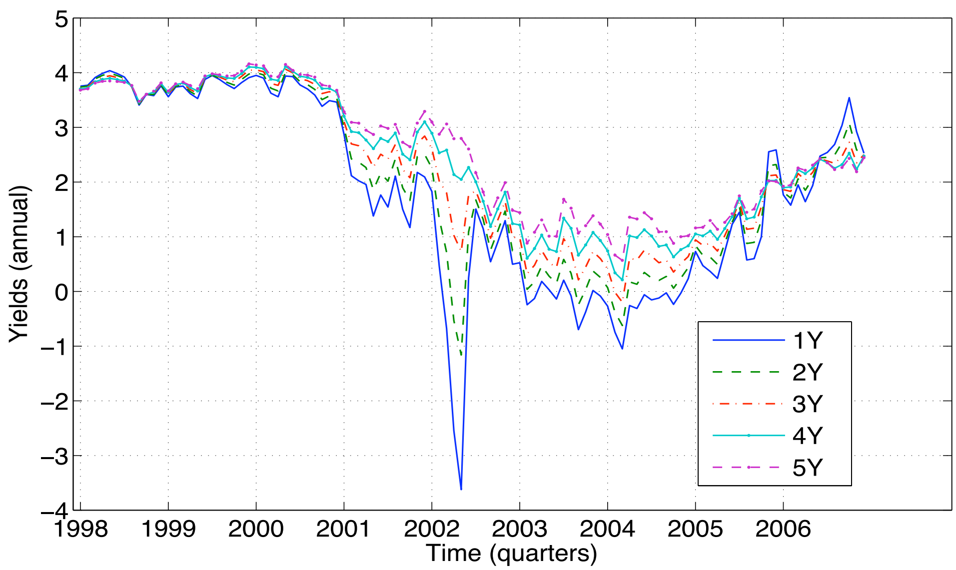

Figure 33: **Time series observed real yields.**

These are the real yields for all 6 maturities as estimated by McCulloch. Note that the periods do not coincide with those considered in our analysis, yet the magnitudes are comparable. Source: www.econ.ohio-state.edu/jhm/ts/ts.html

Table 8: **ML Estimates under no regimes, specification N.** These are the estimates for the state-space with no regimes. Standard errors are in parentheses. Units are quarterly, e.g. the consumption growth under regime 1 is $0.8566 \times 4 = 3.42$ % a year.

Parameter	Estimate	
$\mu_{\Delta c}$	0.8230	
	-	
μ_{π}	0.9267	
	-	
Ω	0.1857	-0.0376
	(0.0180)	(0.0095)
	-0.0376	0.0928
	(0.0095)	(0.0090)
ϕ_x	0.5525	-0.0925
	(0.1707)	(0.0540)
	0.2869	1.0294
	(0.1163)	(0.0346)
$eig(\phi_x)$	0.6168	
	0.9650	
K	0.2384	-0.1226
	(0.0757)	(0.0979)
	0.0921	0.5206
	(0.0489)	(0.0667)

Table 9: **ML Estimates with regimes in Ω, specification H.** These are the estimates for the state-space with one regime with two regime states in the variance-covariance matrix. Standard errors are in parentheses. Time units are quarterly.

Parameter	Estimate	
$\mu_{\Delta c}$	0.8230	
	-	
μ_{π}	0.9267	
	-	
$\Omega(s_{2,t}=1)$	0.1880	-0.0583
	(0.0183)	(0.0153)
	-0.0583	0.1364
	(0.0153)	(0.0220)
$\Omega(s_{2,t}=2)$	0.1880	-0.0124
	(0.0183)	(0.0123)
	-0.0124	0.0378
	(0.0123)	(0.0076)
ϕ_x	0.5641	-0.0882
	(0.1552)	(0.0502)
	0.1848	0.9968
	(0.1065)	(0.0314)
eig(ϕ_x)	0.6058	
	0.9551	
K	0.2296	-0.1304
	(0.0714)	(0.0952)
	0.1238	0.5137
	(0.0456)	(0.0701)
Q	0.9716	0.0284
	(0.0268)	-
	0.0433	0.9567
	-	(0.9714)

Table 10: **ML Estimates, independent regimes in μ_π and Ω, specification I.** These are the estimates for the state-space with regimes in $\mu_\pi(s_{1,t})$ and $\Omega(s_{2,t})$, when $s_{1,t}$ and $s_{2,t}$ are independent. Standard errors are in parentheses. Units are quarterly, e.g. the consumption growth under regime 1 is 0.8566 $\times$ 4 = 3.42 % a year. The means are fixed ex-ante. The mean for consumption growth is the unconditional mean. The variance of the shock impinging consumption growth is set to be the same in each regime state. eig(ϕ_x) are the eigenvalues of ϕ_x.

Parameter	Estimate	
$\mu_{\Delta c}$	0.8230	
	-	
$\mu_\pi(s_{1,t}=1)$	1.1758	
	-	
$\mu_\pi(s_{1,t}=2)$	0.6242	
	-	
$\Omega(s_{2,t}=1)$	0.1836	-0.0587
	(0.0194)	(0.0212)
	-0.0587	0.1624
	(0.0212)	(0.0366)
$\Omega(s_{2,t}=2)$	0.1836	-0.0244
	(0.0194)	(0.0100)
	-0.0244	0.0470
	(0.0100)	(0.0109)
ϕ_x	0.6268	-0.0450
	(0.1780)	(0.0415)
	0.2925	1.0296
	(0.0999)	(0.0240)
eig(ϕ_x)	0.6626	
	0.9938	
K	0.1980	-0.1715
	(0.0958)	(0.0984)
	0.0907	0.5107
	(0.0414)	(0.0670)
Q	0.9761	0.0239
	(0.2536)	-
	0.0258	0.9742
	-	(0.2933)
R	0.9893	0.0107
	(0.3274)	-
	0.1480	0.8520
	-	(0.3208)

Table 11: **ML Estimates with the same regime in μ_π and Ω, specification S.** These are the estimates for the state-space with one regime with two regime states in the mean inflation and variance-covariance matrix. Standard errors are in parentheses. Time units are quarterly.

Parameter	Estimate	
$\mu_{\Delta c}$	0.8738	
	-	
$\mu_\pi(s_{2,t}=1)$	1.1758	
	(0.0993)	
$\mu_\pi(s_{2,t}=2)$	0.6242	
	(0.0965)	
$\Omega(s_{2,t}=1)$	0.1844	-0.0855
	(0.0176)	(0.0242)
	-0.0855	0.1688
	(0.0242)	(0.0394)
$\Omega(s_{2,t}=2)$	0.1844	-0.0239
	(0.0176)	0.0242
	-0.0239	0.0635
	(0.0242)	0.0394
ϕ_x	0.5343	-0.0967
	(0.1770)	0.0641
	0.1773	1.0032
	(0.0915)	0.0278
$eig(\phi_x)$	0.5743	
	0.9632	
K	0.2464	-0.0911
	(0.0730)	0.0958
	0.0603	0.3234
	(0.0423)	0.0531
R	0.9584	0.0416
	(0.0376)	-
	0.0054	0.9946
	-	(0.0054)

Table 12: **ML Estimates for the state-space with regimes in ϕ_x.** The estimates presented are only for ϕ_x.

$\phi_x(s=1)$	-0.0103	-0.1482
	(0.0665)	(0.0643)
	0.2952	0.9430
	(0.0689	(0.0272)
$\phi_x(s=2)$	8.1049	0.2108
	(1.2022)	(0.5841)
	0.7357	0.2458
	(0.3946)	(2.2217)

F Stationarity

In the model section I posit the assumption of needing the eigenvalues ϕ_x strictly less than one. This is necessary for at least two reasons: stationarity and for being able to take the limit of $\sum_i^n \phi_x^i$ as $n \to \infty$, which relate to the expected value of the variables in the model.[62] Figures 34 and 35 present the posterior distributions of the eigenvalues, assuming improper priors. For the component associated to consumption growth, the distribution's support of the eigenvalue is bounded by one. However, the distribution's support of the eigenvalue associated with the component of inflation covers elements above 1. Thus, the assumption of stationarity and being able to take the limit of the aforementioned sums are not innocuous.

G Cross-sectional yields details

Table 14 presents the first four (standardized) moments of the yields for the main specifications of the model, and the data. The behavior of variance (standard deviation) is discussed in the main text.

As for the skewness, all specification are positive. Only specification **I**' capture the negative slope as in the data. Slightly lighter right tails are present for the left part of the curve. Except for specification S the estimates are below. Their magnitudes highlight the importance of the regime in μ_π for the

[62] A more general treament of the subject can be found in Uhlig (1994). He suggests for reasons discuss therein the use of, in the one dimensional case, Jeffrey's prior. In this case it would have to be generalized to two dimensions.

Table 13: **ML Estimates for the state-space with 3 state variables.** These are the estimates for the state-space with no regimes. Standard errors are in parentheses. Units are quarterly, e.g. the consumption growth under regime 1 is $0.8566 \times 4 = 3.42$ % a year. The third variable is labor income growth.

Parameter	Estimate		
$\mu_{\Delta c}$	0.8230		
	-		
μ_{π}	0.9267		
	-		
$\mu_{\Delta L}$	0.5649		
	-		
Ω	0.1811	-0.0342	0.1560
	(0.0183)	(0.0097)	(0.0287)
	-0.0342	0.0869	-0.0612
	(0.0097)	(0.0107)	(0.0206)
	0.1560	-0.0612	0.7135
	(0.0287)	(0.0206)	(0.0699)
ϕ_x	0.0364	-0.1049	0.5496
	(0.7605)	(0.0714)	(0.5128)
	0.7304	1.0463	-0.3559
	(0.8113)	(0.0691)	(0.5482)
	-1.0143	-0.1500	1.1974
	(2.0366)	(0.1451)	(1.2633)
K	0.1686	-0.0307	0.0139
	(0.1095)	(0.0922)	(0.0874)
	0.0951	0.4533	0.0392
	(0.0912)	(0.1177)	(0.0307)
	0.5312	-0.1656	-0.0835
	(0.2919)	(0.1660)	(0.1131)

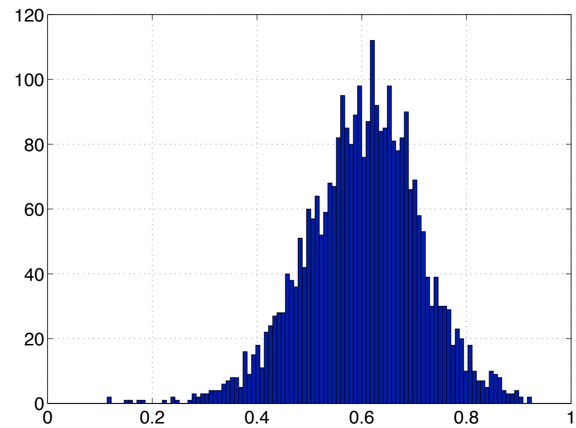

Figure 34: **Posterior distribution of the first eigenvalue of ϕ_x.**

Posterior distributions of the first eigenvalue of ϕ_x. Improper priors were assumed for all parameters except K.

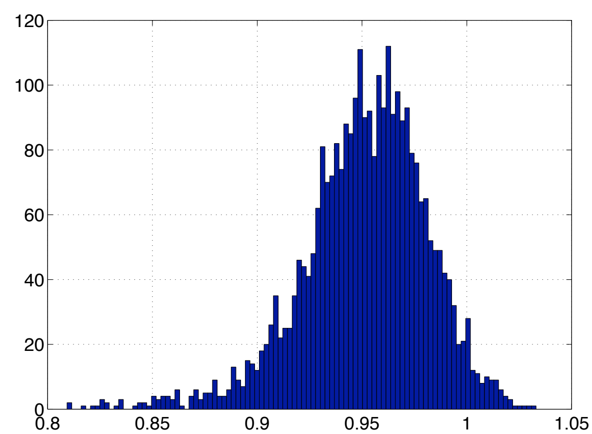

Figure 35: **Posterior distribution of the second eigenvalue of ϕ_x.**

Posterior distributions of the eigenvalues of ϕ_x. Improper priors were assumed for all parameters except K.

implied yields. Overall, elongated right tails for the yields are captured. While the kurtosis is understated in the model, its negative slope is part of the model in specification **I**. Again, hinting the importance of the regime in μ_π. It does well with respect to the autocorrelation, underestimating it slightly. To sum up, provided that only the cross-sectional means of the yields were used for the calibration, in other words the first moment (i.e. the second step of the two-step estimation); the model does an overall satisfactory job. These results support specification **I**.

H The PCA Decomposition for the yields data

The Principal Component Analysis Decomposition for the yields data is:

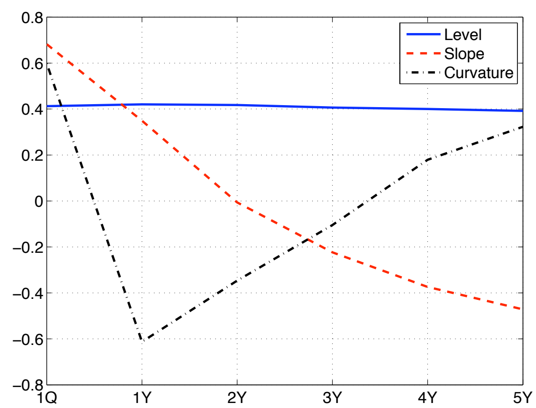

Figure 36: **Principal Components Decomposition for the yields data.**

The plot of the first three principal components for the data are shown. The lines are the “level,” “slope” and “curvature” components.

I Additional predictability regressions

This section presents the additional results related to the tests on yields predictability.

Table 14: **The first four standardized moments.**

The table presents the 1st, 2nd, 3rd and 4th (standarized) moments, and the autocorrelation (1 lag) for the yields implied by the model's specifications and from the data. Recall the notation, specification **I** stands for a the state-space with regime in $s_{1,t}$ in μ_π and $s_{2,t}$ in Ω, **S** stands for the specification which has the same regimes $s_{1,t} = s_{2,t}$ for μ_π and Ω.

	1Q	1Y	2Y	3Y	4Y	5Y
Data						
Mean	5.15	5.56	5.77	5.94	6.07	6.15
Variance	8.53	8.58	8.34	7.93	7.77	7.52
Skewness	1.06	0.82	0.83	0.82	0.83	0.80
Kurtosis	4.46	3.76	3.67	3.61	3.62	3.43
Autocorr.	0.94	0.95	0.96	0.96	0.97	0.97
H						
Mean	5.4624	5.5613	5.7242	5.8900	6.0479	6.1949
Variance	3.3967	2.9050	2.4801	2.1519	1.8768	1.6410
Skewness	0.7794	0.8971	0.9577	0.9763	0.9791	0.9756
Kurtosis	2.9193	3.0971	3.1943	3.2205	3.2197	3.2088
Autocorr.	0.9334	0.9464	0.9521	0.9540	0.9547	0.9548
I'						
Mean	5.6059	5.5699	5.7160	5.8810	6.0425	6.1948
Variance	3.7044	2.3431	1.9899	1.7572	1.5600	1.3849
Skewness	0.8551	0.7137	0.7319	0.7272	0.7158	0.7034
Kurtosis	3.0543	2.7716	2.8098	2.7943	2.7616	2.7258
Autocorr.	0.9298	0.9115	0.8905	0.8763	0.8649	0.8556
I'$_2$						
Mean	5.9470	5.8642	5.8755	5.9297	5.9960	6.0646
Variance	3.6490	2.9896	2.4418	2.0510	1.7493	1.5078
Skewness	1.0507	1.1338	1.1272	1.0936	1.0607	1.0333
Kurtosis	3.8907	3.9881	3.8702	3.7170	3.5878	3.4876
Autocorr.	0.9125	0.9351	0.9471	0.9530	0.9564	0.9586
S						
Mean	5.5245	5.5850	5.7328	5.8958	6.0564	6.2099
Variance	3.8862	3.4205	3.0162	2.6938	2.4167	2.1737
Skewness	1.0686	1.1977	1.2715	1.3112	1.3388	1.3604
Kurtosis	3.2691	3.4997	3.6485	3.7291	3.7835	3.8251
Autocorr.	0.9408	0.9536	0.9582	0.9595	0.9600	0.9601

Table 15: **Holding period returns predictability regressions.** The period is from 1952:2 to 2005:4. Yields source: CRSP. These regressions are for the specifications **S**, $s_{1,t} = s_{2,t}$ and **I**, where $s_{1,t}$ and $s_{2,t}$ are independent. Under the EH $b = 0$. N is in years. $hpr_{t+1}^{(N+1)} - y_t^{(1)} = a + b(f_t^{(N \to N+1)} - y_t^{(1)}) + \epsilon_{t+1}$.

	N	a	$s.e.(a)$	b	$s.e.(b)$	adj. R^2
Data	1	-0.002	0.134	0.919	0.167	0.118
	2	-0.179	0.263	1.232	0.217	0.128
	3	-0.512	0.367	1.507	0.255	0.139
	4	-0.046	0.566	0.932	0.315	0.033
H	1	0.2801	0.0949	0.1426	0.2031	-0.0024
	2	0.5315	0.1931	0.1881	0.2266	-0.0015
	3	0.7446	0.2839	0.2518	0.2404	0.0005
	4	0.9233	0.3639	0.2980	0.2501	0.0021
I	1	0.1271	0.0844	0.5586	0.1866	0.0365
	2	0.3446	0.1959	0.4540	0.2419	0.0121
	3	0.6014	0.3073	0.3862	0.2713	0.0051
	4	0.8225	0.5033	0.3916	0.2852	0.0044
$\mathbf{I}_2$	1	0.0226	0.0688	-0.0120	0.1829	-0.0048
	2	0.1717	0.1256	-0.0120	0.1995	-0.0049
	3	0.3449	0.1772	0.0264	0.2097	-0.0049
	4	0.4918	0.2239	0.0515	0.2196	-0.0048
S	1	0.2847	0.1023	0.0302	0.2527	-0.0047
	2	0.6764	0.2292	-0.0706	0.2963	-0.0046
	3	1.0597	0.3518	-0.0934	0.3166	-0.0045
	4	1.4124	0.5664	-0.1180	0.3315	-0.0044